PRACTICAL OPTICS FOR THE LABORATORY AND WORKSHOP

PRACTICAL OPTICS FOR THE LABORATORY AND WORKSHOP

BY

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WITH A FOREWORD BY

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LONDON: BENN BROTHERS, LIMITED
8 BOUVERIE STREET, E.C.4
1922

Printed in Great Britain
by Turnbull & Spears, Edinburgh

PREFACE

THIS little book has been written primarily as a course of instruction for the student in Practical Optics; and secondly, to deal with the more recent practical applications of Optics for use in the optician's workshop.

The exercises contained in the book are compiled from a series of experiments through which students in the Optical Engineering Department of the Imperial College of Science (South Kensington) usually pass before proceeding to the more advanced optical work of the department. It commences with the quite elementary work and covers a considerable amount of ground, and should, I think, prove a useful laboratory course in "Light" for Colleges and Schools of Science.

The experiments involve as little expensive apparatus as possible, but at the present day satisfactory experimental optics demands somewhat better apparatus than the rather old-time favoured piece of wood and card, and therefore it has been partly the aim in these pages to suggest means of producing such apparatus in the best possible way.

Although some of the devices mentioned are not yet to be found on the market, I have given scale drawings of such things (as, for example, optical benches), all of which have been found thoroughly practical, so that those who have a small workshop available may construct necessary apparatus for themselves.

That the book is not entirely devoted to the use of the student is brought about by the fact that some of the latter chapters—such as Chapter VIII., for example—deal with practical testing of optical instruments, which I

hope may be of some interest and value to the person engaged on work in the testing department of the optician.

All the diagrams are new and by the author.

My thanks are greatly due to Mr L. C. Martin for very valuable assistance and advice during the process of compilation; also to Prof. Cheshire, C.B.E., for kindness in writing the Foreword.

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FOREWORD

THE book which Mr Johnson has written deals primarily with the experimental side of applied optics. Very little will be found in it about caustics, but a great deal about collimators.

The title of the book has the merit of indicating not only its contents, but at the same time giving information as to the way in which the book differs from other books.

Up to the present time there has been an unfortunate want of co-ordination between the practice of the laboratories and that of the workshop, to the distinct disadvantage of both. Each has been in a position to assist the other, but for one reason or another has rarely done so.

Laboratory work, on the other hand, has too often ignored everyday wants. The microscopist, however, who requires simple methods within the compass of his equipment, for the determination of the focal lengths and apertures of his lenses, and the magnifying powers of the various combinations of them, will find all the necessary information given in this book. The owner of a telescope, too, who suspects that only a part of the aperture of his object glass is operative, will now be able to test the matter for himself. He will learn something of the function and importance of that little stop in the erector, the existence of which may not have been suspected.

F. J. CHESHIRE

PRACTICAL OPTICS FOR THE LABORATORY AND WORKSHOP

CHAPTER I

REFLECTION AND REFRACTION OF LIGHT

(*a*) VERIFICATION OF LAWS OF REFLECTION

PLACE a piece of cartridge paper on a drawing board. On this place a mirror, preferably silvered on its "front" surface. A microscope "slip," 3 in. × 1 in., silvered, makes an admirable mirror for the purpose; it should be supported at the back so that it will stand with the silvered face at right angles to the paper. Place two pins A and B (Fig. 1) in front of the mirror in the positions shown. Look at the "images" of the pins, A′ and B′, in the mirror, and adjust two more pins C and D so as to appear in the same straight line as these images. Let the lines through AB and CD intersect on the mirror at O. Draw the normal at O and show that it makes equal angles with the incident ray AB and the reflected ray CD. Repeat this experiment three or four times, using different positions for the pin A, and show that in each case the angle of incidence is equal to the angle of reflection.

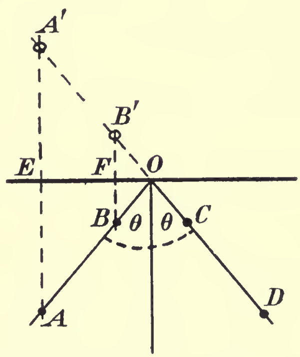


FIG. 1.

The incident ray, the normal to the mirror at the point

of incidence, and the reflected ray all necessarily lie in one plane. Tabulate your values of the angles of incidence and reflection for each ray.

(b) FORMATION OF AN IMAGE BY A PLANE MIRROR

Place a pin P (Fig. 2) in front of the mirror as before, and place the eye in such a position that the lower part of the pin can be seen by reflection. Behind the mirror adjust another pin so that its upper portion appears to be a continuation of the lower portion of the image, for all positions of the eye. The second is then at the image position of the first. Let P and P_1 (Fig. 2) be the positions of the object and image, and let P P_1 cut the mirror XY in O. Prove by actual measurement that $PO = P_1O$ and that the angle POX is a right angle. This will show that the image on the normal to the mirror is as far behind the mirror as the object is in front.

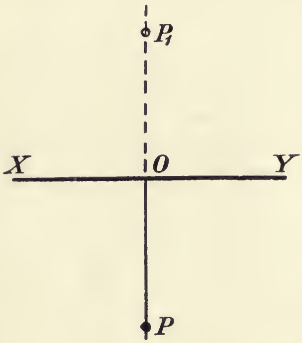


FIG. 2.

In Fig. 1 produce DC back to A′ and measure AE, A′E, BF, and B′F.

(c) LAWS OF REFRACTION

First Law.—The incident and refracted rays, and the normal at the point of incidence, all lie in the same plane.

Second Law.—The ratio of the sines of the angles of incidence and refraction for the two media in question is constant. (See Fig. 3.)

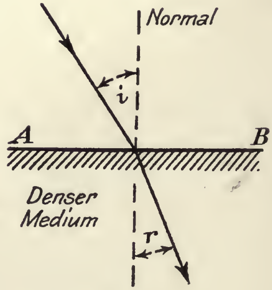


FIG. 3.

Explanation of the phenomenon of refraction by the "wave" theory of light.

The "wave theory" is now a fully established fact, and refraction is very easily made clear by considering it on these lines.

Let AB (Fig. 4) be the bounding line between two media, and suppose the lower half to be the denser medium. Let the velocity of light in the upper medium be v, and in the lower v_1. Let Cc, Dd, and Ee be three rays in an oblique parallel beam of light, and CDE the

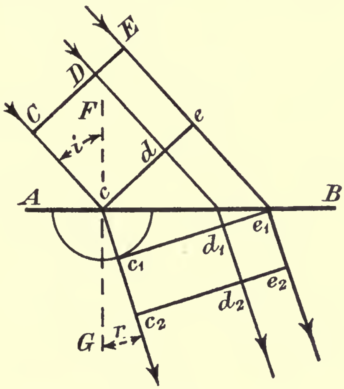


Fig. 4.

wave front at any instant. This will advance parallel to itself until it reaches cde. The ray Cc then enters a different medium, and its velocity is changed from v to v_1. Consequently, whilst the ray Ee is travelling from e to e_1, Cc will move through a distance $\left(\frac{v_1}{v} \times ee_1\right)$. With c as centre and $\left(\frac{v_1}{v} \times ee_1\right)$ as radius, describe a semi-circle in the lower medium. From e_1 draw a tangent to this semi-circle, touching it at c_1. Join cc_1. Then cc_1 will be the new direction of the ray Cc; c_1e_1 will be the new wave front; and the disturbance at e will travel to e_1 in the same period of time "t" that the disturbance at c travels to c_1.

Now FG is a normal to AB at the point c, and the angle the beam was making with this normal was CcF. But having undergone this change of direction (*i.e.* refraction) in the denser medium the angle is now c_2cG.

The ratio of the sines of these two angles is constant whatever incidence is given to Cc, and this ratio is known as the "*refractive index*" between the two media.

Refractive Index is usually denoted by the letter "n," so that the above may be written $n=\frac{\sin i}{\sin r}$.

It is also easily shown from Fig. 4 that the "refractive index" is also the ratio of the velocities of light in the two media :—

For $n=\frac{\sin i}{\sin r}$.

Now $\sin$ "i" $=\sin ece_1=\frac{ee_1}{e_1c}$,

and $\sin$ "r" $=\sin ce_1c_1=\frac{cc_1}{e_1c}$;

$$\therefore\ n=\frac{ee_1}{cc_1}.$$

But $ee_1=vt$.
and $cc_1=v_1t$;

$$\therefore\ n=\frac{v}{v_1}.$$

(*c*) EXPERIMENTAL WORK FOR $\frac{\sin i}{\sin r}$=A CONSTANT.

Place a block of glass (about 4 in. × 3 in. × 1 in.) with parallel sides on a piece of drawing paper, and draw two fine lines along the two edges AB and DC (Fig. 5). Place a pin P in the position shown in contact with the edge of the block, and arrange a series of pins $P_1P_2P_3P_4$ on the circumference of a circle whose centre is P. The radius of this circle should be about 3 in. This gives a series of incident rays P_1P, P_2P, P_3P, P_4P. Determine the paths of these rays through the glass by placing against the side DC pins P_5, P_6, P_7, P_8, which appear to be in the

same straight line as $P_1P_2P_3P_4$ respectively. Remove the block and join PP_5, PP_6, PP_7, PP_8. Also draw a normal PN to the surface AB.

Measure with a protractor the angle of incidence and the angle of refraction for each ray and show that $\frac{\sin \text{"}i\text{"}}{\sin \text{"}r\text{"}}$ is constant.

This ratio is the "refractive index" of the material. It may be determined graphically from the figure by completing the circle $P_1P_2P_3P_4$, thus cutting the refracted rays. Draw perpendiculars to the normal PN from the

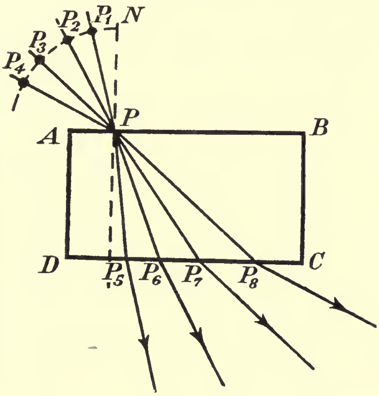


FIG. 5.

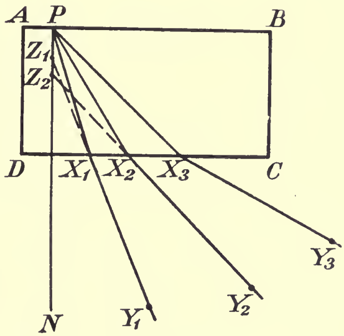


FIG. 6.

points at which an incident ray and a corresponding refracted ray cut the circle. The ratio of these perpendicular lengths gives the required result.

Second Method.—Place the glass block as before on the drawing paper, and draw fine lines along the edges AB and CD (Fig. 6). Place a pin at P in contact with the edge AB. Insert other pins $X_1X_2X_3$ on the other edge DC in the positions shown. On looking through the glass, place further pins $Y_1Y_2Y_3$ so that they appear in the same straight line as X_1P, X_2P, X_3P respectively. Remove the block, and draw a normal PN to AB at P. Join Y_1X_1 and produce it back to cut the normal in Z_1. Produce each other ray back in the same manner.

Show by actual measurement that $\frac{PX_1}{Z_1X_1}=\frac{PX_2}{Z_2X_2}$, and so on, is constant. This ratio is the "refractive index" of the glass.

Explanation of foregoing.—Consider Fig. 7. PX_1 and Y_1X_1 are the same rays as lettered thus in Fig. 6. Draw a second normal OR at X_1. Then Y_1X_1R = angle of incidence, and PX_1O = angle of refraction.

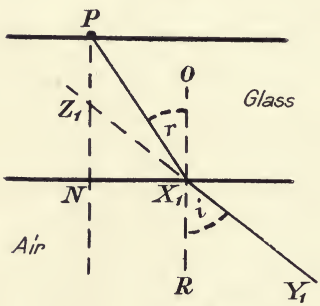


Fig. 7.

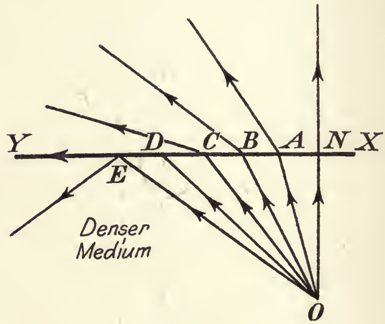


Fig. 8.

But PN and OR are parallel.

So that $\angle Y_1X_1R = \angle NZ_1X_1$

and $\angle PX_1O = \angle NPX_1$.

Now $n=\frac{\sin \text{“}i\text{”}}{\sin \text{“}r\text{”}}$.

$$\text{Therefore } n=\frac{\sin NZ_1X_1}{\sin NPX_1}=\frac{\dfrac{NX_1}{Z_1X_1}}{\dfrac{NX_1}{PX_1}}=\frac{PX_1}{Z_1X_1}.$$

(*d*) TOTAL INTERNAL REFLECTION

Total internal reflection is dependent on refraction. Fig. 8 shows how a series of rays coming from a point "O" in a denser medium than air are refracted at the bounding surface XY (*e.g.* a stone in a pool of water).

Let "n" be the refractive index, which in this case will be less than unity. Now, for any angle of incidence "i" (measured inside the denser medium) the angle of

refraction "r" is calculated from the formula $n = \frac{\sin i}{\sin r}$. So that, in this case, "r" is always greater than "i."

As "i" increases the refracted rays get nearer and nearer the surface, until a position is reached such as ODY, where "r" $= 90°$. Now the sine of $90°$ is unity, and no angle has a sine greater than unity, so that for our formula $\left(n = \frac{\sin i}{\sin r}\right)$ to give any real value for "r," $\frac{\sin i}{n}$ must be equal to, or less than, unity. Thus, for a refracted ray to be formed, the greatest value of "i" is when $\sin n = i$. This angle is called the "*critical angle.*"

The question then arises—What happens to the incident

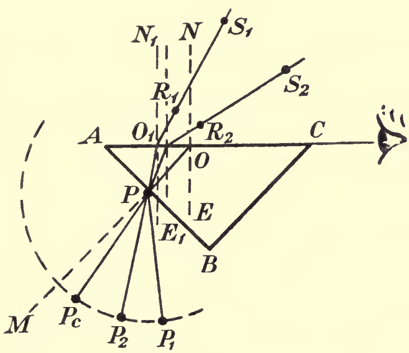


FIG. 9.

rays when they meet XY beyond OD, as at E, making greater angles with the normal? In this case no light is refracted, but all which falls on the surface is reflected back into the first medium. This phenomenon is termed "total internal reflection." At any angle, however, a certain amount of internal reflection takes place.

Experiment I.—Place a right-angled prism on a piece of drawing paper on a drawing board. The prism should be a large one, preferably with the hypotenuse surface about 4 in. long. Draw fine lines round the three faces of the prism. Place a pin P, as shown in Fig. 9, in contact with the surface AB. With P as centre and PP_1 as radius

(about 4 in.) describe a semi-circle about AB. Place a second P_1 in some such position as indicated, and on looking at the hypotenuse face AC, insert further pins R_1 and S_1 so that they appear in the same straight line as PP_1. This will give the refracted angle in air $N_1O_1R_1$ corresponding to the incident angle PO_1E_1 in the glass. Move the pin P_1 into another position P_2 so that the angle PO_1E_1 is increased, and insert the pins R_2 and S_2. In this way move P_1 continually towards M until the eye can only just see the two images of P and P_1 in line with the perpendicular edge C of the surface AC. This will give the last ray in the glass that is able to get outside the bounding surface AC. Remove the prism, and draw a normal PM. Join P_cP. P_c is this last position of P_1. Measure the angle MPP with a protractor. With the refractive index of the prism given, calculate the refracted angle, and draw in PO, making this angle with the normal PM. This is the "critical angle" for this particular glass.

Experiment II.—Total Internal Reflection.—Replace the prism on the paper as in Fig. 9, and place P in the same

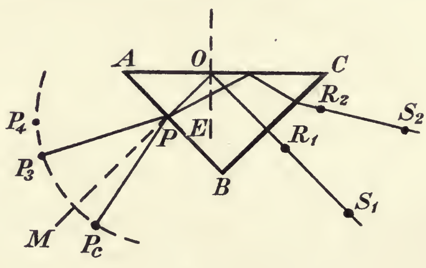


Fig. 10.

position as before. Move P_c further towards M so as to make the angle POE just greater than the critical angle. Then place the eye so as to look in the face BC, and position the pins R_1 and S_1 (Fig. 10) so as to appear in the same straight line as PP_c (Fig. 10). You will now notice that as soon as the angle POE is made greater than the critical angle, the ray is totally reflected at the face AC.

Repeat the experiment for other positions of P_c as indicated at P_3 and P_4, and show in each case that the ray undergoes "total internal reflection."

(*e*) **"SMITH'S" RAY PLOTTER** (*Trans. Opt. Soc.*, 1919-20, vol. xxi., No. 3).

In the case of all graphic experiments in connection with refraction, it is continually necessary to draw refracted rays at the bounding surfaces of media. These angles have, in an ordinary way, to be calculated from the formula $n = \frac{\sin \text{“}i\text{”}}{\sin \text{“}r\text{”}}$, and then drawn out with a protractor, which, if a number of surfaces are involved, becomes very laborious and also occupies a great deal of time. Therefore it is of great advantage if these angles can be obtained readily and easily; this "ray plotter" here described gives a means of doing this, and in a very simple manner.

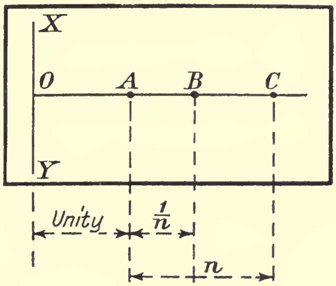


FIG. 11.

Procure a piece of thin sheet celluloid, 6 in. long by $2\frac{1}{2}$ in. wide (see Fig. 11). On it scratch a fine straight line OC, with a marking point; also a line XY at right angles to this at O, as shown in the figure. Mark off distances OA = 2 in., AB = $1\frac{1}{3}$ in., and AC = 3 in. The relation between these distances is dependent on the refractive index, but this is explained later. All glasses, of course, have not the same refractive index; but for graphic experiments such as would be done in the laboratory the refractive index of glass would probably be taken as approximately 1·50. And this is the value on which the above figures are based. If a particular type of glass, of known refractive index, is in question, then of course the above values will differ, but in every case OA must

be equal to unity, AB equal to $\frac{1}{n}$, and AC equal to "n" in some convenient unit.

At the points A, B and C drill three very small holes, just sufficient in diameter to take the point of a pin. The "ray plotter" is now complete.

How to use it.—Suppose we wish to determine the direction of the refracted ray EM (Fig. 12) in a piece of glass corresponding to an incident ray VE. Place the "ray plotter" on the paper, so that the line XY lies on the bounding line of the two media PR, and O of the "ray plotter" coincides with E on the paper.

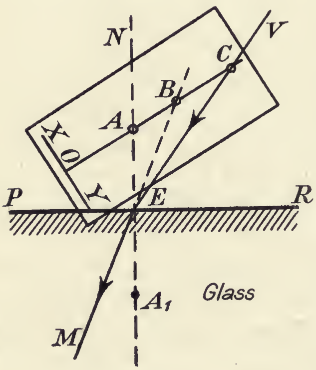


Fig. 12.

Insert a pin through the hole A and revolve the celluloid until the hole C comes directly over the incident ray VE. Prick the paper through the hole B; remove the "plotter," and join this point to E. This line produced EM gives the refracted ray.

If the ray in the rarer medium is required to be traced from the ray in the denser medium, the method of procedure is very similar, with the exception that when the celluloid is revolved about the point A,* the point B must be made to coincide with the *refracted ray* in the denser medium. The paper is then pricked through the hole C and this point joined to E and produced. This gives the ray in the rarer medium.

Proof of Method.—Let VE (Fig. 13 *a* and *b*) represent the incident ray in the rarer medium incident at the point E of the denser medium, and EM the corresponding refracted ray. EA is the radius of a circle and equal to unity. Draw AC (Fig. 13 *a*) perpendicular to EA, and in Fig. 13 *b* draw it obliquely. Where the refracted ray

* In this case the hole A will be in the position A_1 (Fig. 12).

EM cuts AC, describe a circle with radius AB; and where VE produced cuts AC, describe a circle with radius AC.

It is required to show, that for $\dfrac{\sin \mathrm{NEV}}{\sin \mathrm{MEA}}$ to be constant, AC must equal "n" and AB $\dfrac{1}{n}$ (where n = the refractive index).

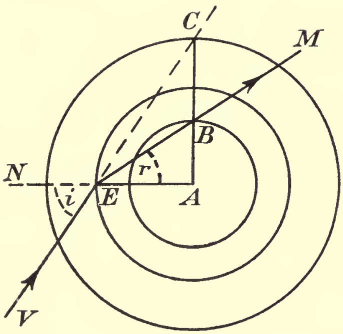


Fig. 13 (*a*).

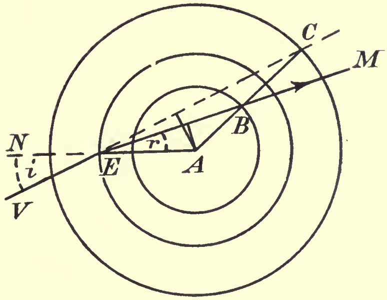


Fig. 13 (*b*).

Now the triangles AEC and AEB are similar (from the ratio of sides and a common angle);

so that $\angle \mathrm{AEC} = \angle \mathrm{EBA}$

and $\angle \mathrm{ECA} = \angle \mathrm{AEB}$

Then, in triangle AEC,

$$n = \frac{\sin i}{\sin r} = \frac{\sin \mathrm{AEC}}{\sin \mathrm{ECA}} = \frac{\dfrac{\mathrm{AC}}{\mathrm{EC}}}{\dfrac{\mathrm{EA}}{\mathrm{EC}}} = \frac{\mathrm{AC}}{\mathrm{EA}}.$$

But EA is unity.

Therefore $\mathrm{AC} = n$.

Similarly, in triangle AEB,

$$n = \frac{\sin i}{\sin r} = \frac{\sin \mathrm{EBA}}{\sin \mathrm{AEB}} = \frac{\dfrac{\mathrm{EA}}{\mathrm{EB}}}{\dfrac{\mathrm{AB}}{\mathrm{EB}}} = \frac{\mathrm{EA}}{\mathrm{AB}}.$$

But EA is unity.

Therefore $\mathrm{AB} = \dfrac{1}{n}$.

(*f*) PATH OF RAYS THROUGH A 60° PRISM

On a sheet of drawing paper on a drawing board place a 60° glass prism. A large prism should be used for this experiment, preferably about a 3 in. face and refractive index about 1·52. Draw fine lines along the two sides AB and BC (Fig. 14). Remove the prism for a moment, and at the mid-point P_1 of AB draw a normal N_1D. At P_1 set off a line P_1P_2 at 40° to the normal with a protractor. Place two pins, one at P_1 and the other at P_2; then put the prism back into its former position. On looking in the face BC of the prism arrange two more pins P_3 and P_4 so that they appear in the same straight line as P_1P_2. The images of P_1 and P_2 will be fringed with colour owing to dispersion, but this will not interfere with the positioning of the pins P_3 and P_4. Remove the prism, join P_4P_3 and let it meet the surface BC in F. At this point draw a second normal N_2D. Measure the angle of emergence P_4FN_2 corresponding to the angle of incidence $N_1P_1P_2$ (which was 40°). Produce P_4F and P_2P_1 and let them meet at "O." Then the angle P_4OM is the *deviation* produced by the prism.

Increase the angle of incidence $N_1P_1P_2$ by 5° and repeat the experiment, and so on until $N_1P_1P_2$ is as large as possible. In each case measure the angle of emergence from the prism and the deviation, and tabulate the results as follows :—

Angle of Incidence.	Angle of Emergence.	Deviation.
40°		
45°		
50°		
55°		
60°		
65°		
70°		
75°		

On a piece of squared paper then plot two curves, one

showing the relationship between the angle of incidence and the angle of emergence, and the other between the angle of incidence and deviation. Plot incidence angles in a horizontal direction and emergence and deviation in

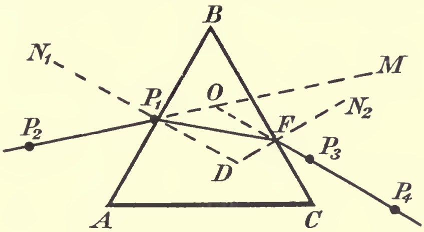


FIG. 14.

a vertical direction. Fig. 15 shows the type of graphs obtained.

Observe from the curves you obtain that there is a position where the "deviation" is at a minimum. The angle of incidence should be noted for this position; also

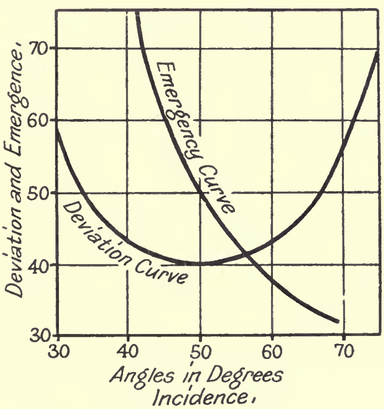


FIG. 15.

by reference to the "emergence angle curve" the corresponding emergence angle will be obtained. If your curves are plotted correctly these two angles will be found the same.

This shows that when the incident and emergent angles are equal, the deviation of the prism is at its minimum.

To determine the position of minimum deviation.—Draw a straight line P_1P_2 (Fig. 14) on a piece of drawing paper, and place two pins in the positions P_1P_2. Place the 60° prism as indicated so that P_1 touches the face AB. Look into the face BC and place the eye so that the images of P_1 and P_2 appear in the same straight line. Now rotate the prism slowly, first in one direction and then in the other, moving the eye the whole time so that the two images always appear in the same straight line. A position will be noticed when the two images, moving in one direction, suddenly become stationary, and commence to move in the opposite direction. This stationary position of the images is the position of "minimum deviation" for the prism. Insert two pins P_3 and P_4 so that they appear in the same straight line as P_1 and P_2. Remove the prism, join P_4 P_3, and show that the angles of incidence and emergence are equal.

(*g*) PATH OF RAYS THROUGH A 45° PRISM

(i) Place a 45° prism ABC (Fig. 16) on a piece of drawing paper (the prism should be large, from 3 in.

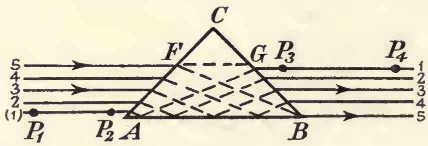


FIG. 16.

to 4 in. hypotenuse face). Mark fine pencil lines round the three faces. Remove the prism for a moment, and draw five lines to the left of AB parallel to the hypotenuse AC. Number these lines 1 to 5, and replace the prism. On line "1" place two pins P_1 and P_2. Look into the face BC, and insert further pins P_3 and P_4 so that they appear in the same straight line as P_1P_2. Do this for all the five incident rays, and number each corresponding emergent ray. Remove the prism, and draw in the path of the rays through the prism, remembering the laws of refraction and reflection.

Note that "internal reflection" takes place at the face AB, also that an "*up and down*" reversal of the object takes place. It will also be seen that there is a limit to the "useful aperture" of the prism when used in this way; for after No. 5 ray in the figure has got through, the portion FCG is no longer useful, as no more rays above F can get through the face BC. The figure AFGB is called an erecting prism.

(ii) Place the prism on a fresh piece of paper and draw fine lines round the edges as before. Remove the prism and draw a series of parallel lines at right angles to AB

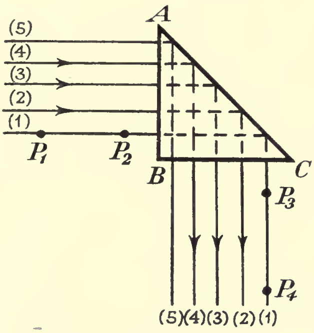


FIG. 17.

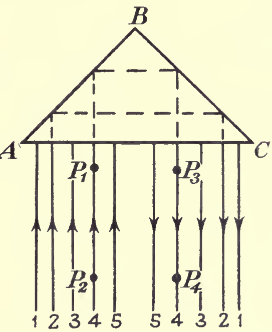


FIG. 18.

(Fig. 17). Number these lines and replace the prism. Insert pins P_1 and P_2 on the No. 1 line, and looking in the face BC place P_3 and P_4 so as to be in the same straight line with the images of P_1 and P_2. Do the same for rays Nos. 2, 3, 4 and 5. Note that as the rays are incident normal to the face AB, no deviation takes place at the refracting surfaces, but that total reflection takes place at the face AC. Also, observe that there is a "right and left" reversal of the object in this case.

(iii) Determine as before, by means of the pin method of ray-tracing, the paths of rays when they are incident on the hypotenuse face AC (see Fig. 18). Note in this case that a total internal reflection takes place at both surfaces AB and BC, and also that there is again a right

and left reversal of the object. Right-angled prisms are used in this last manner in prismatic binoculars.

(*h*) CONSTANT DEVIATION PRISMS

(i) There are two special types of prism which should be noted in connection with the work of this chapter.

They are at present in everyday use and involve principles dealt with here. The first of these is illustrated in Fig. 19, and is known as a Pentagonal Prism; these prisms are used to a very great extent on military and naval "range-finders." The figure shows the direction and path of the

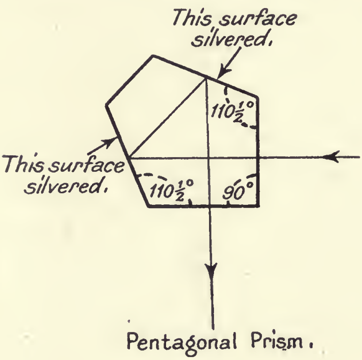


Fig. 19.

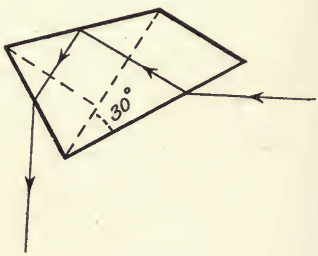


Fig. 20.

rays through the prism, and, as will be seen, "internal reflection" takes place at the silvered surfaces. This is not the same kind of internal reflection that has been dealt with before in this chapter, as that is dependent on the critical angle; in this case it is essential that the two surfaces of the pentagonal prism indicated should be silvered. The importance of this prism, however, lies in the fact that the "deviation" between the incident and emergent rays always remains constant, and also that this deviation is 90°. If this type of prism is available in the laboratory, the above points should be proved by ray-tracing with pins.

(ii) The second type of prism is illustrated in Fig. 20, and is used a great deal in connection with spectrometers.

This prism also gives "constant deviation" between the incident and emergent rays. The path of the rays are indicated in the figure, and, as will be seen, they undergo two refractions and one total internal reflection. The prism is all one piece of glass, but the dotted lines indicate how it may be considered as built up from two 30° prisms and one 45° prism.

If the laboratory has this type of prism, rays should be traced through it by pin methods.

CHAPTER II

MIRRORS AND LENSES (OPTICAL BENCH EXPERIMENTS)

(*a*) DESCRIPTION OF OPTICAL BENCH

AN optical bench of the type here described is very convenient in a laboratory. Its combined simplicity and accuracy make it invaluable for both instructional and

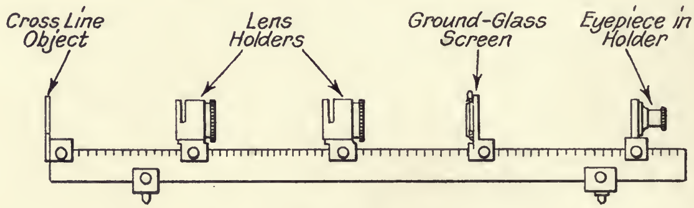


FIG. 21.

commercial work. Fig. 21 shows the general appearance of the bench, and, as will be seen, it consists of a Chesterman steel metre rule supported in a vertical plane, along which all other necessary fittings slide. These fittings are all

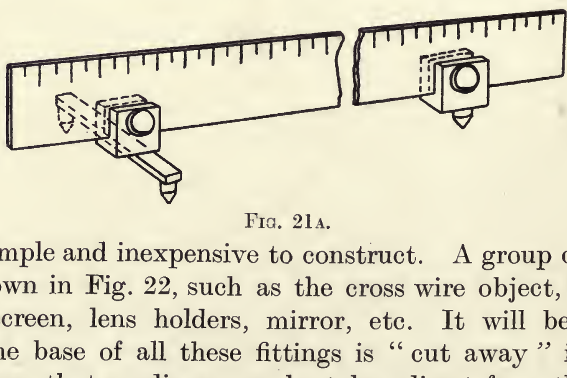


FIG. 21A.

very simple and inexpensive to construct. A group of these are shown in Fig. 22, such as the cross wire object, ground glass screen, lens holders, mirror, etc. It will be noted that the base of all these fittings is "cut away" in such a manner that readings may be taken direct from the steel

rule without any appreciable error being introduced. Where more accurate results are necessary a "correction rod" may be employed. The lens holders are designed to

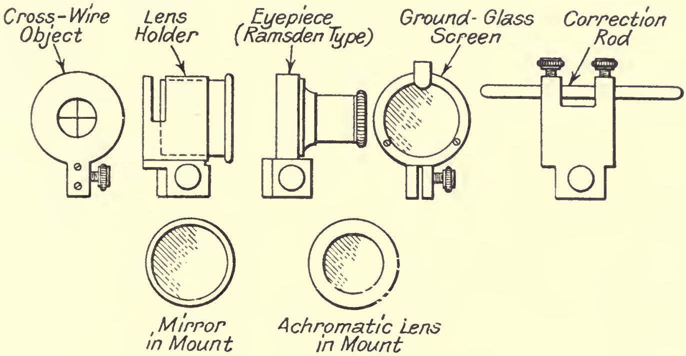


FIG. 22.

carry lenses from any ordinary spectacle trial case, so that for experimental work a large range of lenses may be obtained.

The fittings that support the steel rule in a vertical

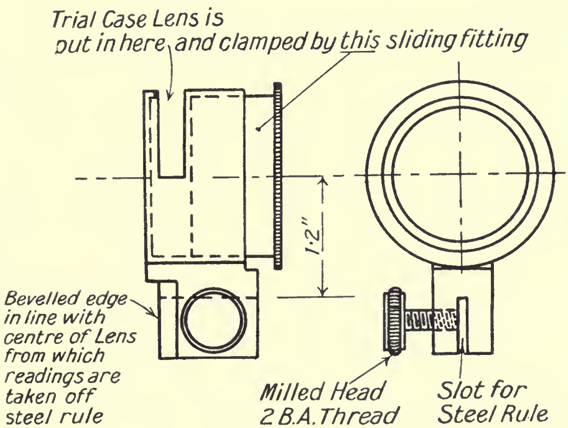


FIG. 23.

position are also shown in Fig. 21A. These are adaptable not only to the metre rule but to shorter lengths, such as a foot rule, when such experiments only involve small ranges. Scale drawings of the lens holders and ground

glass screen holders are shown in Figs. 23 and 24. Fr these and Fig. 22 a general idea of all the fittings may obtained. (See Article by Prof. Cheshire in *Trans. O Soc.*, vol. xxii., No. 2.)

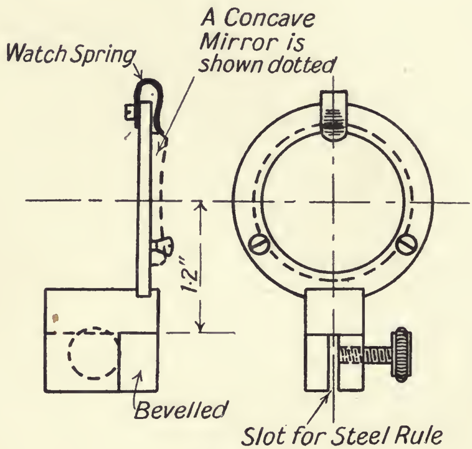


Fig. 24.

(b) MEASUREMENT OF THE RADIUS OF CURVATURE OF A CONCAVE MIRROR OR CONCAVE LENS SURFACE

The concave mirror * provided for this experiment should be held in one of the optical bench fittings so that the

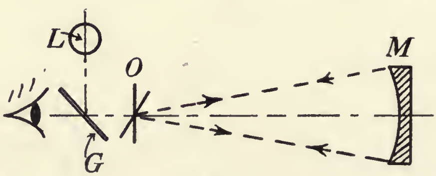


Fig. 25.

" pole " of the mirror is in the same plane as the edge of the mount from which readings are taken (see Fig. 24).

Arrange on the optical bench the cross-wire object and the mirror whose curvature is required. Place a plane

* These mirrors may be made very simply by silvering the surfaces of a convex and concave lens from an ordinary spectacle lens "Trial Case," and mounting them with the silvered surface "outwards."

ss reflector G (micro cover slip) diagonally, as shown in . 25, so that light from a lamp L (an electric lamp rosted" or covered with a piece of tissue paper) uminates the cross-wire object O. Place the eye in the osition shown and an "image" of the cross-wires will seen near the "real" cross-wires reflected from the rface M. It is at once evident that if the "image" ud "real" cross-wires are in the same plane the dis- nce MO will be the radius of curvature of the mirror, for all rays diverging from O will return back along their original paths, and therefore they must strike the mirror normally (the normal to a spherical surface at any particular point is its radius). The method of ensuring that "image" and object are in the same plane is by employing the parallax method. By moving the head from side to side the "image" of and "real" cross-wires will appear to move together when the mirror is in its correct position; if, however, the "image" does not appear to move as fast as the "real" cross-wires the plane of the image will lie behind the plane of the object, and *vice versâ*. As an alternative, the "image" may be focussed directly on the white surface at the back of the cross-line object.

When the curvature of a *concave lens surface* is required, exactly the same procedure is employed, with the exception that the surface not under test must be covered in some manner in order to prevent stray light being reflected back. If this back surface of the lens is covered with a thin layer of "plasticine," this serves the purpose very well. A piece of blotting paper stuck to the back surface with vaseline does equally well. The "image" of the cross-wires will not be so bright as when a silvered surface is used, but sufficiently bright for taking measurements.

(*c*) RADIUS OF CURVATURE OF A CONVEX MIRROR OR A CONVEX LENS SURFACE

Arrange the apparatus on the metre "optical bench" as shown in Fig. 26. O is the cross-line object at the

end of the steel rule. A is an achromatic lens * (held in one of the lens holder fittings) which forms an image of the cross-wires on the ground glass screen S. G is a plane glass reflector which illuminates the object from a lamp at L.

Determine carefully the reading, on the optical bench, of the ground glass screen S when the image is sharply in focus. Interpose the convex mirror † to be tested M, in the position indicated, and adjust its position until on viewing the object as in the last experiment, the plane of the image "thrown back" by the mirror M is coincident with the plane of the "object." This is done by the parallax method as before or by focussing the

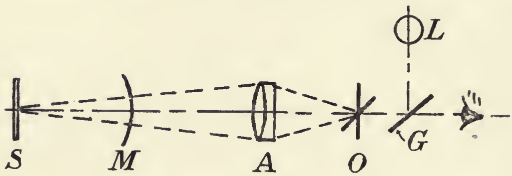


Fig. 26.

image direct on the white surface at the back of the cross-line object. The reading of the mirror is then taken, and the distance SM is the radius of curvature of the surface. For, in order that the "rays" leaving O and A should retrace their paths after reflection from the mirror and form an "image" at O, they must strike the mirror "normally," and this is only the case when the distance SM is the radius of curvature of the surface. A number of independent readings should be taken for the position of M and the mean obtained.

For the determination of the radius of curvature of a convex lens surface, the same method is adopted, the back surface of the lens being covered by some such method as mentioned in the previous experiment.

* An "achromatic" lens is used to prevent undue dispersion of the light, which would otherwise arise with a "single" lens.

† See footnote on page 32.

Curvature (Introductory)

The curvature of a circle may be defined as being equal to the reciprocal of its radius.

$$\frac{\mathrm{CD}}{\mathrm{DA}}=\frac{\mathrm{DA}}{\mathrm{CE}-\mathrm{CD}};$$

and when the angle DOA is small

$\frac{\mathrm{CD}}{\mathrm{DA}}=\frac{\mathrm{DA}}{2r}$ (very nearly, r being the radius of circle).

Whence $\mathrm{CD} \propto \frac{1}{r}$.

Thus the length CD, known as the "sagitta" (trigonometrically the versed sine of the angle DOA) is a measure of the curvature of the arc ACB. This fact is the foundation of the curvature method.

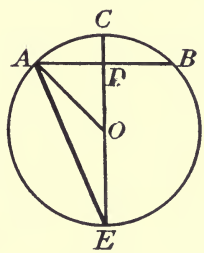


FIG. 27.

So that, light waves as they reach a lens or mirror from a point source at a distance "u" have a curvature equal to $\frac{1}{u}$, and this curvature has a *negative* sign when the waves are "convex-fronted" and thus expanding from a focus; and a positive sign when they are "concave-fronted" and thus contracting to a focus. Similarly the curvature imparted or "*impressed*" by a positive lens of focal length "f" is equal to $+\frac{1}{f}$, whilst in the case of a negative lens it is equal to $-\frac{1}{f}$.

The curvature "impressed" upon a plane-fronted wave by a mirror or lens is defined as its "focal power."

This power is impressed upon all waves acted upon, no matter at what distance the object may be. Thus the curvature of each wave, as it emerges from a lens, or it may be reflected by a mirror, is equal to the curvature of the incident wave added to the curvature impressed by the lens or mirror. In other words, final curvature equals initial curvature + that impressed.

If u = distance of object to lens
v = ,, image ,,
and f = focal length of the lens,
then $\frac{1}{v}=\frac{1}{u}+\frac{1}{f}$.

(*d*) **FOCAL LENGTH OF A CONVEX LENS (THIN)**

(i) Place a 5D lens from the "trial case" in one of the lens holders on the metre optical bench. Direct the optical bench at the furthest bright object that can be seen—for instance, a street lamp, or an electric lamp placed in a long corridor,—the distance should not be less than 50 yards. Place also on the bench a ground glass screen in its holder and receive an "image" of the distant lamp produced by the lens on this. The difference between the readings of the lens holder and ground glass screen holder will give the "focal length" of the lens. Make a number of independent settings and measure the distance in each case. See how nearly any one measurement is likely to be correct.

(ii) After having used a distant object, use an object comparatively near to the lens. This method involves the use of the formula $\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$, where "$f$" is the focal length of the lens, "u" the distance between the object and the lens, and "v" the distance between the "image" and the lens. Due respect must be made to the use of signs when employing this formula, and it should be remembered that divergent light is always reckoned as possessing negative curvature, whilst convergent light is positive. Set up the cross-line object O (Fig. 28) at one end of the optical bench and illuminate it with a lamp. Place the 5D lens L (in holder) at a distance of about 45 cms. from the object and receive an image of the cross-lines on the ground glass screen. Take a number of independent readings for the position of this screen. Measure the distance "u" (object to lens), in this case it will be a negative value. Also measure "v" (image

to lens), this will be a positive curvature. From these values calculate the result for "f."

Move the lens to another position (say 55 cms. from the object) and repeat the experiment.

(iii) *Auto-collimation Method.*—It will be seen from Fig. 29 that if light diverging from the object O is rendered

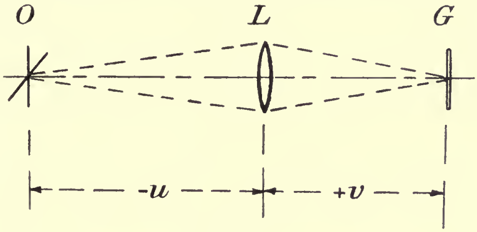


FIG. 28.

parallel by the lens L, reflected back by a mirror M, and again brought to a focus by the lens, the distance OL will be the focal length of the lens. Set up the object O at the end of the bench as before and illuminate it; place the lens about 20 cms. from the object, and further along the bench place the mirror M in position. Care-

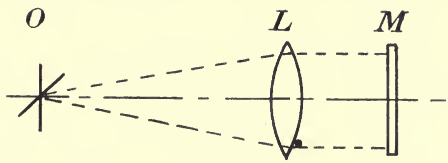


FIG. 29.

fully adjust the lens holder until an " image " of the object is sharply focussed on the whitened back of the object.* Measure the focal length OL. Take a number of independent readings for the position of L. Take a mean value of your results for each method and compare their results.

(*e*) FOCAL LENGTH OF THIN CONCAVE LENSES

Set up the cross-wire object O (Fig. 30) at one end of the optical bench, and form an image of this by means

* The mirror may require tilting slightly.

of the achromatic lens A on the ground glass screen S_1. Place a -3D lens from the trial case in one of the lens holders and insert this in the path of the convergent beam at L. Move the screen until the image is again focussed, as at S_2. The image produced at S_1 by the lens A serves as the object for the negative lens, so that the distance LS_1 is "u" and is positive, while the distance LS_2 is "v" and is also positive. Using the formula $\left(\frac{1}{f}=\frac{1}{v}-\frac{1}{u}\right)$ as before, the focal length of the negative lens may be determined. All values of readings taken from the "bench" should be the "mean" of a number of

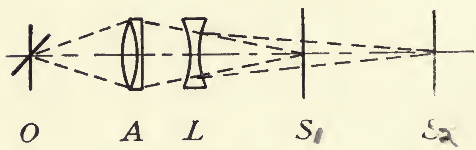


Fig. 30.

independent settings. Move the negative lens L to a fresh position and repeat the experiment.

(*f*) RELATION BETWEEN SIZE OF IMAGE AND FOCAL LENGTH OF A LENS

Set up the metre optical bench with a lens holder mounted on it. Arrange at the zero end of the steel rule a piece of ground glass screen ($4\frac{1}{4}$ in. $\times$ $3\frac{1}{4}$ in.) in a vertical plane so that the ground surface lies flush with the end of the rule. As far away as it is possible to arrange, set up two light sources at the same height as the optical bench. Make the distance apart of these two lamps about 6 or 8 feet, so that they subtend a small angle at the lens. In the lens holder place, in turn, lenses from the trial case ranging from a $+2$D to $+12$D, varying by 1D every time. In every case measure the distance between the centres of the two images produced on the ground glass screen. This is most easily done by laying a short millimetre rule on the ground glass and observing

with a watchmaker's eyeglass. The position of the lens holder on the optical bench when the images are sharply in focus on the screen must be taken for each individual lens. This will give the focus of the lens (approximately). Tabulate the values for the distance apart of the images and the corresponding focal lengths for each lens, and plot these values on squared paper. On the same piece of paper plot the reciprocal of the focal length against the distance apart of the images. Write down the meaning of your graphs thus obtained.

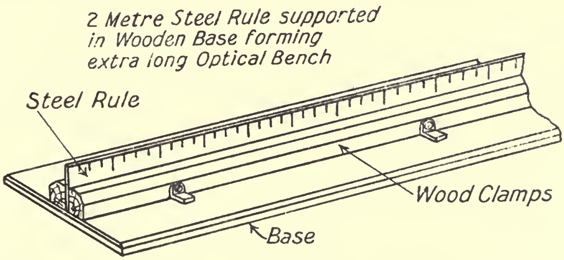


Fig. 31.

(*g*) THE RELATION BETWEEN THE CONJUGATE DISTANCES AND CURVATURES FOR THIN POSITIVE AND NEGATIVE IMAGING LENSES

For this experiment the optical bench is employed, but in place of the metre steel rule a two-metre steel rule is used, as a larger working range is necessary. A two-metre steel rule can be obtained from Messrs Chesterman (of Sheffield), and is preferable, but it is possible to use two one-metre rules placed end on to one another. In either case it is better to mount them in a wooden base, a portion of which is shown in Fig. 31, so that any tendency of the steel to bend is prevented.

Experiment.—To obtain and plot the curve showing the relationship between the position of the object and its image.

Positive Lens (convex)

Case I.—A " real " object moving up from the left (see Fig. 32) to the first focal point of the lens, *i.e.* the curvature

of the incident light-waves is negative and varies from O to $-\frac{1}{f}$ (where "f" is the focal length of the lens).

In this case the image is always real, and can therefore be focussed on a ground-glass screen.

Place the cross-wire object at the extreme left-hand end of the bench, and illuminate it by means of a lamp placed behind it. Place a 5D lens L (Fig. 32) in one of the optical bench lens holders, and adjust its position on the bench so that the distance IO (I is the "image plane" and recorded by the ground-glass screen) is the maximum obtainable under the conditions.

Adjust the screen I so that the image is sharply focussed. Then measure the distance $LO = u$ $(-)$ and $IL = v$ $(+)$.

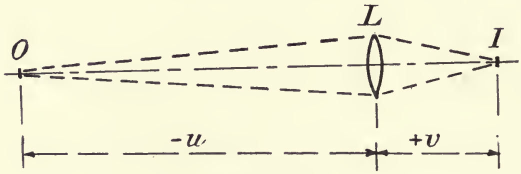


Fig. 32.

Move L a short distance (say 5 cms.) nearer to O and repeat the experiment. In this way obtain a series of pairs of values for "u" and "v." Plot these values on a piece of squared paper, remembering that when the incident waves are diverging, "u" is plotted negative; when converging, positive. A graph should be obtained similar to the one shown in the top left hand quadrant of Fig. 38. On a second sheet of squared paper plot the curvatures $\frac{1}{u}$ and $\frac{1}{v}$ for the same experimental data (see Fig. 40).

Case II. (see Fig. 33).—A "real" object moving from the first focus to the lens; *i.e.* the curvature of the incident waves is negative and varies from $-\frac{1}{f}$ to $-\infty$.

In this case the image is virtual and cannot therefore be focussed on a ground-glass screen. So that, for this part of the experiment the optical system is arranged

as shown in Fig. 33. First set up a simple telescope by employing the achromatic lens A and an eyepiece (these should be standard fittings for the optical bench). Focus this telescope for "parallel light" on some very distant object, and situate it near the middle of the bench. Place

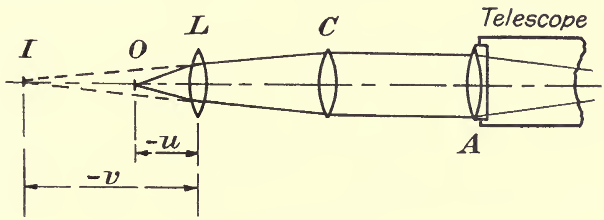


Fig. 33.

the +5D lens L with its holder near the object (*i.e.* within 20 cms.), and the beam now passing out from L will be divergent. By inserting a further lens C (from the trial case) of known focal length, say a +2D, in this divergent beam, the light will be rendered parallel, so that looking through the telescope a virtual image I of O will be seen; this image is situated at the principal focus of C. Therefore LO $=u$, and $v=$focal length of C − CL.

Change the position of L and repeat. In this way obtain a series of pairs of values for "u" and "v," giving that

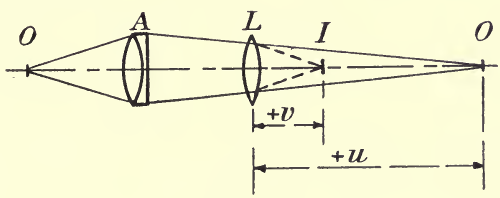


Fig. 34.

portion of the curve between the limits $u=-f$ and $u=0$. This curve will be seen in the lower left-hand quadrant of Fig. 38. Also plot the corresponding curvatu e curve (see Fig. 40).

Case III. (see Fig. 34).—A virtual object moving from the lens to the right, *i.e.* the curvature of the incident is positive and varies from $+\infty$ to O.

Place the achromatic lens A to the left of the bench and adjust it so as to give an image O′ of the cross-wires near the right-hand end of the bench. Insert the +5D lens L in the path of the convergent beam and receive the image I on the ground glass screen. Then $LO' = u$ and $LI = v$. Obtain a series of pairs of values for "u" and "v" as before, commencing with "u" as about 110 cms. and moving L step by step until "u" is about 5 cms. Plot these values as a continuation of the last curve (see Fig. 38, top right-hand quadrant), also the curvature graph as for Cases I. and II. (see Fig. 40).

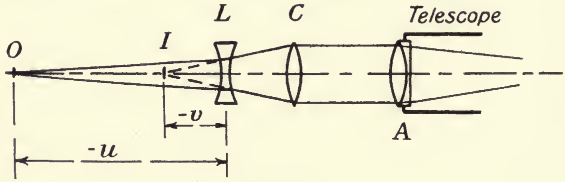


FIG. 35.

Curves for Negative Lens (concave)

Case I. (see Fig. 35).—A "real" object moving up to the lens from the left, *i.e.* the curvature of the incident light is negative, and varies from O to $-\infty$.

Place the cross-wire object at the extreme left-hand end of the bench. Arrange the telescope with the achromatic lens and eyepiece (as for Case II. of the positive lens) at the right-hand end. In a lens holder place the −5D lens L, and make its distance from the object (*i.e.* −"u") 100 cms. Between this lens and the telescope insert an auxiliary positive lens of known focal length (from the trial case), about a +2D, and adjust it until the object is brought sharply into focus when looking through the telescope. Then $OL = -u$, and focal length of $C - CL = v$. Move the position of L and repeat. Make a series of pairs of values for "v" and "u" as before, and plot them on a fresh piece of squared paper (see Fig. 39, both on left-hand quadrant), also the curvature values, $\frac{1}{v}$ and $\frac{1}{u}$ (see Fig. 41).

Case II. (see Fig. 36).—A "virtual" object moving from the lens to the second focus of the lens, *i.e.* the curvature of the incident light is positive and varies from $+\infty$ to $\frac{1}{f}$. In this case the image is real and can be focussed on a ground-glass screen.

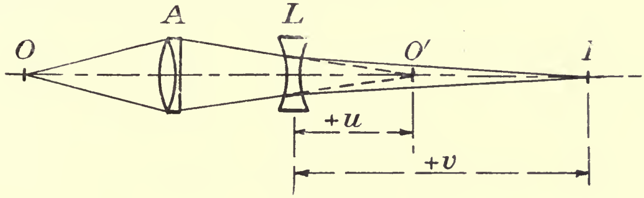


Fig. 36.

Place the object at the left-hand end of the bench and arrange the achromatic lens A to form an image O′ of O at about the middle of the bench. Place the -5D lens L in the convergent beam about 3 cms. to the left of O′ and adjust the screen until the image is again sharply in focus. Then $O'L = +u$, and $IL = +v$. Move L a short distance, say 1 cm., and repeat the experiment. In this way obtain as before a series of pairs of values for "u" and "v." Plot these as a continua-

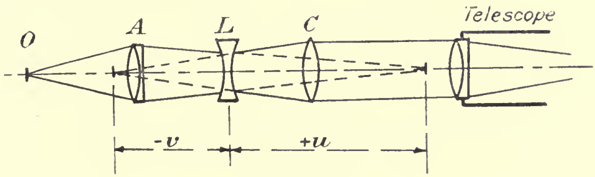


Fig. 37.

tion of the curve for the last case (see Fig. 39, top right-hand quadrant). Also plot the corresponding $\frac{1}{u}$ and $\frac{1}{v}$ curve (see Fig. 41).

Case III. (see Fig. 37).—A "virtual" object moving from the first focus of the lens to the right, *i.e.* the curvature of the incident light is positive and varies from $\frac{1}{f}$ to O.

Retain the same positions of the object O, the achromatic lens A, and consequently the image O′. The -5D lens L should then be placed a short distance to the right of A, so as to make LO′$=u$ as large as possible. The image I now being virtual, obtain its position by means of the telescope and auxiliary lens as before. Then LO′$=+u$, and LI$=+v$.

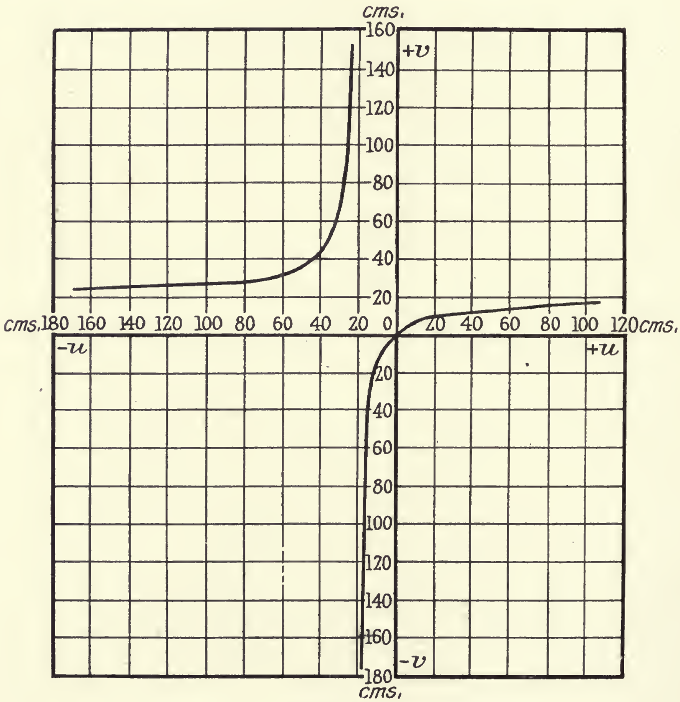


Fig. 38.

Move L further to the right by, say, 5 cms., and repeat the experiment. Obtain, as before, a series of pairs of values for "u" and "v" and plot them (see Fig 39, right-hand lower quadrant). Also curvature graphs $\frac{1}{u}$ and $\frac{1}{v}$ (see Fig. 41).*

* In all the above experiments the sharpness of the "images" may be improved by using a yellow "colour filter" in front of the object, in order to cut out the blue rays.

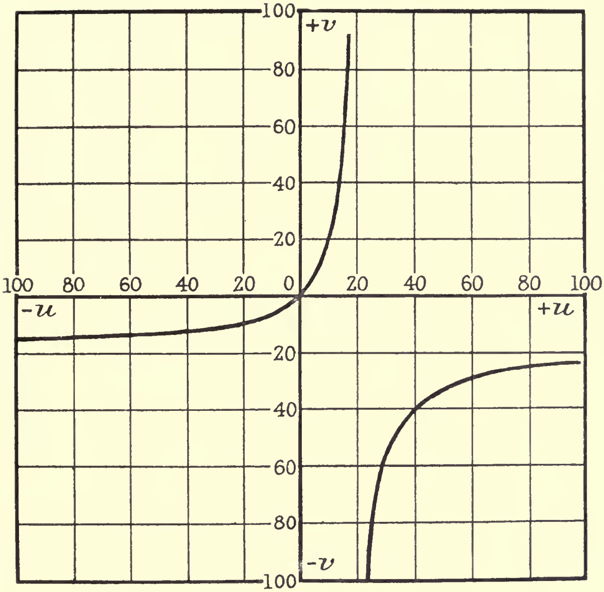


Fig. 39.

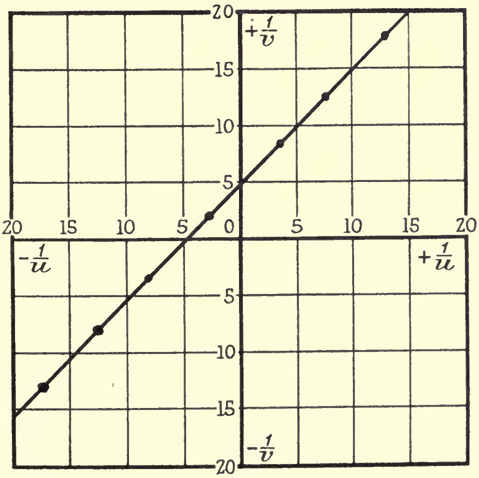


Fig. 40.

These six experiments give the full data for plotting the curves shown in Figs. 38, 39, 40 and 41.

(*h*) **SIMPLE TELESCOPE**

The experiment consists in setting up a simple astronomical or inverting telescope and taking measurements in connection with the "system," and then repeating the measurements for a Galilean telescope.

Astronomical.—Use a metre optical bench for the experiment. At the left-hand end place a positive lens (from the trial case) of fairly long focal length, *e.g.* a +2D, in one of the lens holders. Receive an image

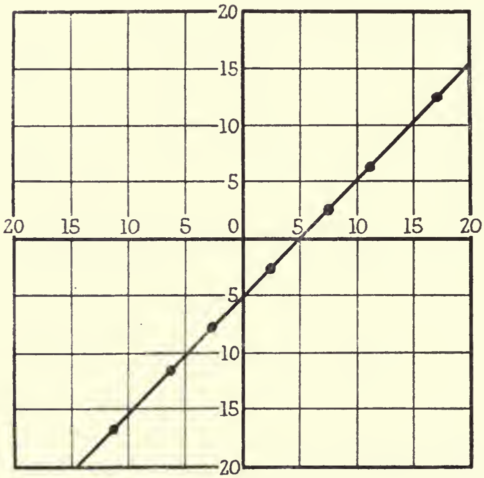


FIG. 41.

of some very distant object (such as a lamp), produced by the lens on the ground glass screen (in its holder). Place in a second holder, and on the *other* side of the ground glass screen, a short focal length positive lens, such as a +12D. Turn the optical bench completely round, and again focus the distant object on to the ground glass screen by adjusting the position of this lens holder. Then remove the ground -glass screen, and look at the distant object through the system of the two lenses. This is a simple form of inverting or astronomical telescope; the +2D lens would be known as the object glass, while the +12D is the eyepiece (see Fig. 42).

Observe that :

(i) the image is larger than the object as seen directly, *i.e.* it subtends a greater angle at the eye.

(ii) the image is inverted and reversed.

(iii) that the edge of the field of view is indefinite and ill-defined optically.

Measure the distances, off the optical bench, from the object glass to the image, and from the eye lens to the image, and compare these values with the nominal focal

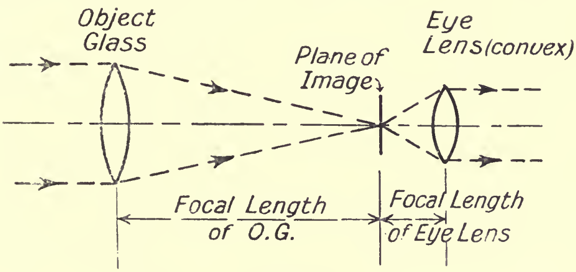


Fig. 42.

lengths of the two lenses as given by the focal power engraved on the lens ring $\left(\text{focal length} = \frac{100}{\text{Power}} \text{ in cms.}\right)$.

Repeat these measurements with two other telescopes made up from different pairs of lenses, and tabulate the results, as follows :

Distance of O.G. from Image.	Nominal Focal Length of O.G.	Distance of Eye Lens from Image.	Nominal Focal Length of Eye Lens.	Object-Glass to Eye Lens.	Sum of Nominal Focal Lengths.

Observe that the distance apart of the lenses when the telescope is focussed for parallel light is equal to the sum

of the focal lengths. In this condition the telescope is said to be in "normal" or "afocal" adjustment.

Find the position with a ground glass screen of the image of the O.G. aperture projected by the eye lens. This image is variously known as the Ramsden circle, the eye-ring, or the exit-pupil. Note that for comfortable vision this image must fall on the pupil of the eye of the observer.

Field of View.—Note that only when the eye is placed in the plane of the "eye-ring" will the whole available field appear fairly well defined.

Measurement of Field of View

Direct Determination.—Place two candles at the far end of the room and adjust their distance apart until the images of the flames as seen in the telescope are just simultaneously visible one in either edge of the field of view. Measure the distance from the O.G. of the telescope to the mid-point between the two candles L, and let the distance apart of the candles be D. Then the field of view of the telescope in degrees θ is: $\tan\theta = \frac{L}{D}$ (approx., as long as the angle is small).

(i) *Magnifying Power.*—Use the telescope with the $+2D$ lens as "object glass," and the $+12D$ lens as eye lens. Observe through it with one eye a distant vertical scale pinned to a wall (the divisions should be about 10 in. apart), whilst with the other eye the scale is seen directly. Note how many divisions of the scale, seen by the unaided eye, are covered by a single division as seen through the telescope. The number of divisions thus seen in the space of one magnified division is equal to the magnifying power of the telescope. Compare this result with the calculated value of the magnifying power obtained by dividing the focal length of the O.G. by that of the eye lens.

(ii) *Determine the Magnifying Power from the diameters of the entrance and exit pupils.*—Illuminate the O.G. with diffused light, by placing a frosted lamp close to it. Place

a millimetre scale on glass * in one of the optical bench fittings, and receive an image of the O.G. aperture projected by the eye lens on to it. Measure the size of this image with the scale. Also measure the diameter of the O.G. (with a pair of dividers). Then, the magnifying power $= \dfrac{\text{diameter of entrance pupil}}{\text{diameter of exit pupil}}$.

Draw a sketch to illustrate how the "magnified" image is formed in the astronomical telescope.

Galilean Telescope.—Set up, on the metre optical bench as before, a +2D lens in a holder at about the middle of the bench. Receive an image of a distant lamp pro-

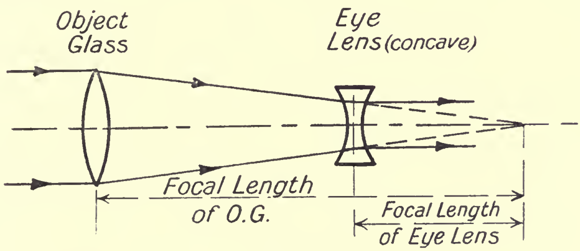


FIG. 43.

duced by this lens on a ground glass screen. Put a −12D lens in a holder, and place it on the bench *between* the O.G. and the ground glass screen, but nearer the latter. Observe the distant object through the telescope and adjust the position of the eye lens until the object is sharply in focus. This is now a simple form of Galilean telescope (see Fig. 43).

Observe that :

(i) the image is larger than the object as seen directly, *i.e.* it subtends a greater angle at the eye.

(ii) the image is erect and not reversed, as in the case of the simple astronomical telescope.

(iii) the edge of the field of view is indefinite and ill-defined optically.

* These glass scales may be obtained from Messrs Rheinberg & Co., 23 The Avenue, Brondesbury Park, N.W.6.

Repeat the same experiments with the Galilean telescope as mentioned before with the astronomical telescope, and tabulate all the results.

Draw a sketch to illustrate how the magnified image is formed in a Galilean telescope.

(*i*) THE SIMPLE COMPOUND MICROSCOPE

Place the cross-wire object at one end of the metre optical bench, and the ground glass screen (in its holder) about three-quarters of the way to the other end of the bench. Place a short focus lens, say a +10D trial case lens, in one of the lens holders, and adjust its position, not far from the cross-wires, so that a "magnified" image

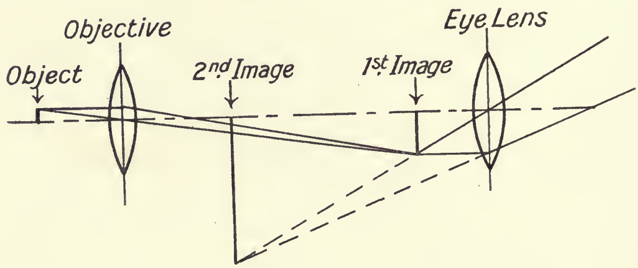


Fig. 44.

of the latter is given on the screen. Now, take another fairly strong lens, say a +12D, and mount it in a holder on the other side of the ground glass. Adjust the position of this lens until a very distant object is focussed sharply on the screen, but do not move the ground glass screen. The screen may now be moved and the "aerial" image of the cross-wires observed by looking through this second lens.

This is now a simple form of compound microscope (see Fig. 44).

Measuring the "First Magnification."—This is the ratio of the sizes of the first "real" image produced by the first lens, and the object. Place one of the millimetre scales on glass (paragraph (*h*) of this chapter) against the cross-wire object, and a second centimetre scale at

the position where the "aerial" image is formed by the first lens. See how many divisions of this latter scale cover one division of the magnified image; then determine how many cover two magnified divisions, and so on; thus obtain the "first magnification" of the microscope.

Magnifying Power.—Compare the image of a definite number of divisions of the millimetre scale against the cross-wires, as seen through the microscope, with the same number of divisions on a second scale as seen directly with the other eye at a distance of about 10 in. (the "near point" of the eye). Of course, in making this comparison the microscope must be so focussed that the image of the first scale seen through it is formed apparently at a distance of 10 in., and not at infinity, as was the case before.

Again, see how many divisions of the scale seen directly cover one division of the "magnified" scale, and thus obtain the magnifying power. Draw a sketch to illustrate the formation of the magnified images.

Stops.—Try the effect on the image of cutting down the aperture of the front lens:

First to half the diameter.

Second to quarter the diameter.

Carefully describe the effects produced.

Note that the eye must be placed at the "eye-ring" in order that the whole available field shall be fairly well defined.

CHAPTER III

PHOTOMETRY

FOR the theory of Photometry, text-books should be consulted; a good book on this subject is "Illumination and Photometry," by Wickenden.

Introduction.—The basis of all photometric comparisons between light sources is the law that the intensity of light given out by a source varies inversely as the square of its distance. Suppose a luminous point is giving out light in all directions. It is at once obvious that a sphere, whose centre is the luminous point, will be equally illuminated over its entire interior surface. Let "r" be the radius of any particular sphere, then the area covered $=4\pi r^2$.

Suppose L to be the amount of light emitted by the source per second, then the illumination per unit area $=\frac{L}{4\pi r^2}$.

Thus, the illumination at a given distance from a source of light is inversely proportional to the square of the distance. This law is known as the "Inverse Square Law."

A photometer is a means of measuring the relative luminosities of two light sources by the simple expedient of estimating (with the human eye) the quality of two illuminations thrown on a white screen by the two light sources, and by being able to measure accurately the distance between the screen and the lamps, when equality of illumination due to the two lamps is secured. A standard lamp, such as the Vernon-Harcourt Pentane Lamp or the Hefner Alteneck (see text-books), of known candle-power, may be employed as one of the sources of light,

so that the other may be determined in candle-power. This is obtained from the distances "r" and "r_1," measured from the lamps to the screen when the intensities are "matched," for :

If L is the amount of light emitted by one source per sec., and L^1 is the amount emitted by the other, the illuminations per unit area are $\frac{L}{4\pi r^2}$ and $\frac{L^1}{4\pi r_1^2}$ respectively, but these intensities are "matched" or equal, so that :

$$\frac{L}{L^1} = \frac{r_1^2}{r^2}.$$

If L is the standard of known candle-power, by simply measuring "r" and "r_1" the candle-power of the lamp L^1 under test can be obtained. This is the principle on which all photometers are based.

There are many types of bench photometers, but they all involve the necessity of having a darkened room with the walls painted with a "dull black" varnish, in order to stop any reflections, which would otherwise interfere with the results obtained. A small room should be chosen for the purpose, and a good coat of "dull black" spirit varnish given to the walls.

(*a*) "RICHIE" PRISM PHOTOMETER

This photometer consists of two right-angled prisms of about $\frac{3}{4}$ in. face "balsamed" to the polished side of a piece of ordinary focussing screen, so that the two edges of the prisms (see Fig. 45) touch one another. The piece of ground glass with the prisms now attached is mounted in a vertical position in a small wooden framework (see Fig. 46), with a circular aperture for observation. This can then be mounted on one of the metal fittings so as to slide on the "two-metre" optical bench referred to in Fig. 31. When in use, the light sources, *i.e.* the "standard"

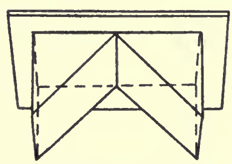


FIG. 45.

and the lamp being tested, are placed in the same straight line as the steel rule, preferably at each end, and on observing through the circular aperture of the photometer the ground glass screen will be seen to be illuminated,

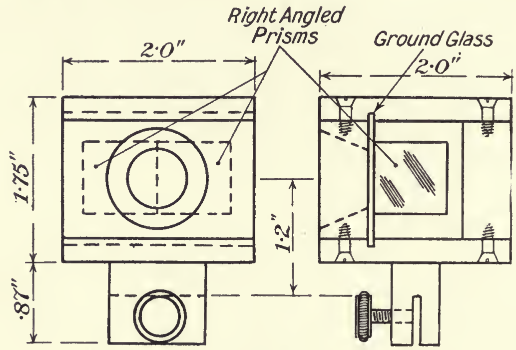


Fig. 46.

half the aperture from one source of light and the other half from the second source. This is brought about by the manner in which the prisms are arranged (Fig. 45), so that total internal reflection takes place and illuminates the ground glass.

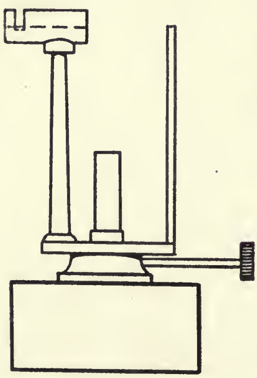


Fig. 47.

Thus by moving the photometer backwards and forwards along the optical bench, a position will be found where the two halves of the aperture are of equal intensity, and the distances between the photometer and the two lamps obtained. If necessary the " standard " and the lamp under test may be mounted on fittings to slide on the optical bench, in order to attain more accurate results.

A very good standard lamp for early experiments in photometry is the Hefner-Alteneck. This lamp is shown in Fig. 47; the height of the flame can be adjusted and measured; and this standard may be trusted to within

about 2 per cent., provided that correction for pressure and humidity of the air have been made.

Experiment.—Set up the photometer just described on the " two-metre " optical bench, and place at one end the standard " Hefner " lamp, and at the other end, or nearly so, place an electric lamp (preferably " carbon " filament) of about 16 candle-power. Arrange the electrical connections for this lamp as shown in Fig. 48, so

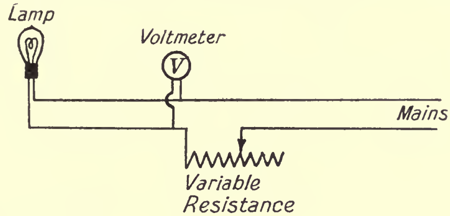


FIG. 48.

that there is a variable resistance * in the circuit and also a voltmeter across the lamp terminals. In this way the candle-power of the lamp can be varied and in each case determined by the photometer, whilst a corresponding voltage from the voltmeter may be read off in each case. About ten different candle-powers of the lamp should be taken, and a graph plotted showing the relationship between the voltage and candle-power.

(*b*) RUMFORD PHOTOMETER

The principle of this type of photometer is shown in Fig. 49. A circular rod " A " is placed a short distance in front of a white screen BC, the two sources of light to be compared are placed at S_1 and S_2 so that shadows of the rod fall on the screen at " *ab* " and " *ac* " respectively,

* A very simple and convenient type of variable resistance is made by using a large (12 in. × 12 in.) photographic developing dish which has in it a solution of water and a few drops of acid hypo. The wires from the " main," and to the lamp, should have small plates soldered to them, and then put into the solution. By varying the distance apart of these two plates, thus immersed, a very fine adjustment for procuring difference in voltage is obtained. The amount of acid hypo put in is small, but is found on trial; the solution should be well stirred.

and are coincident at "*a*." The distances of S_1 and S_2 are adjusted, usually, by allowing one of them to move

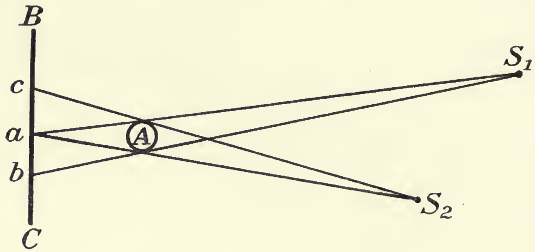


FIG. 49.

along a divided scale, until the two shadows appear equally dark. Then, as before, the intensities of the two lamps will be proportional to the inverse square of their distances from the screen.

Experiment.—Set up a two-metre optical bench as referred to in Fig. 31, in a darkened room, and on one of the sliding fittings for this bench mount an electric lamp (about 16 candle-power carbon filament). At the zero end of the bench place the white screen. This may be constructed in the following manner: cut cut a block of wood to the shape shown in Fig. 50, and on its front face attach a small strip of brass about $\frac{3}{4}$ in. wide, and then cover the rest of the wood on this face with black velvet. Take a piece of "magnesium ribbon" about 6 in. long and ignite one end; hold the block immediately above the flame and allow the brass strip to be well coated with the oxide thus produced. The velvet should be covered with a piece of card which has an aperture cut in it to allow the brass plate to project through. This gives a very good screen for photometric work.

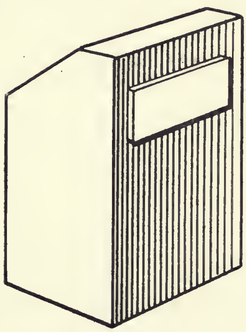


FIG. 50.

The standard (Hefner) lamp should then be set up on

the table in some such position as shown at S_2 in Fig. 49, and the electric lamp " wired " as before (see Fig. 48). A small circular rod (such as a pencil) should be mounted near the screen and its distance adjusted until the edges of the two shadows produced by the lamps are coincident. Having set the " standard lamp " at a known distance from the screen (measured with a steel tape), move the electric lamp along the optical bench until the two shadows on the screen appear equally dark. Take the distance given by the optical bench between the screen and this lamp and obtain its candle-power from the formula given before ; also note the voltage from the voltmeter. Repeat this a number of times, in each case altering the voltage by separating the two wires in the hypo solution (see variable resistance in last experiment), and thus plot a graph showing the relationship between candle-power and voltage.

(c) PHOTOPED

The construction of this photometer is illustrated in Fig. 51. It consists of two tubes A and B, about $1\frac{1}{2}$ in. diameter, sliding one inside the other. Inside the tube B is a metal " stop " with a rectangular aperture

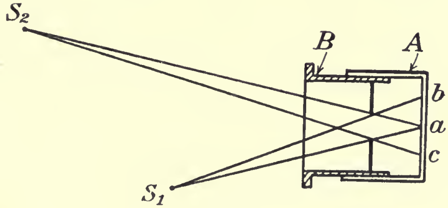


Fig. 51.

($\frac{1}{4}$ in. $\times$ 1 in.) cut in it. At the end of the tube A is attached a translucent screen, such as a piece of oiled or greased paper. The two sources of light to be compared are placed at S_1 and S_2, and the light proceeding through the aperture in B illuminates the screen attached to A with two rectangular patches of light, as shown by " *ab* "

and "*ac*" in the figure, the edges of which are made to coincide at "*a*" by the adjustment of the tube B either towards or away from the screen.

By arranging the two patches of light to appear equally bright, the intensities of the lamps may be obtained as before.

Experiment.—Precisely the same experiment as performed with the other two types of photometer may be done with the "Photoped," by setting it up at the end of the two-meter optical bench and carrying out the same instructions. This type of photometer is used a great deal in actual practice by "gas referees."

LUMMER-BRODHUN PHOTOMETER

The Lummer-Brodhun type is a rather better and more accurate photometer. The instrument is shown in Fig. 52. Two screens of magnesium oxide (as applied

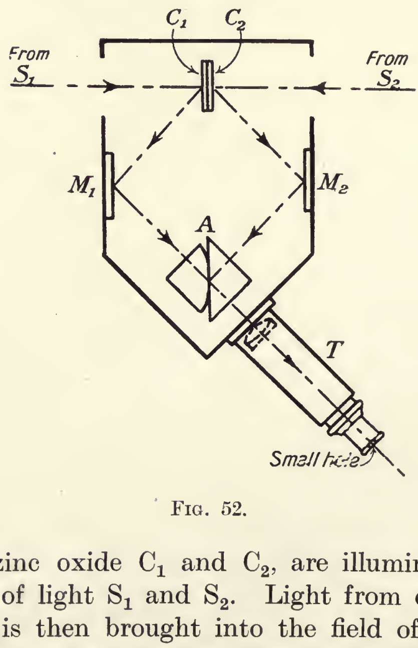


Fig. 52.

before), or zinc oxide C_1 and C_2, are illuminated by the two sources of light S_1 and S_2. Light from each of these two screens is then brought into the field of view of the telescope T by means of two mirrors M_1 and M_2 and a

Lummer-Brodhun cube A. Such a cube is shown in Fig. 53, and consists of two right-angled prisms which are put in "optical contact" over a small circular area in the centre of their hypotenuse faces. The remainder of the face of the prism 1 is ground away as indicated; this allows light from both mirrors M_1 and M_2 to enter the telescope. The appearance seen is that of two concentric circles of light, the centre patch of light coming

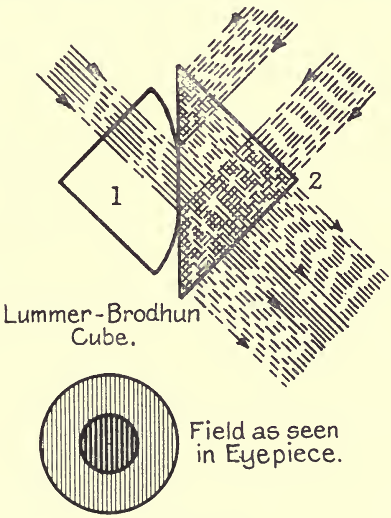


Fig. 53.

from the source S_1 and the outer ring of light from S_2; with such an arrangement difference in equality of the intensities is very easily detected. When in use the light sources are usually kept stationary and the photometer moved until the intensities of the two parts of the field are equal; when in this position the bounding line between the two parts will not be visible. This photometer may be adapted very conveniently to a rather larger type of photometer bench. In some instruments other methods of dividing the field are adopted.

LARGE PHOTOMETER BENCHES

The application of the "steel-rule optical bench" for use as a photometer bench is quite suitable for early and introductory experiments in photometry, but for more advanced work and general use a larger type of mounting for photometers is desirable.

For this purpose a double-lined track (similar to that shown in Fig. 54) is usually employed, which supports the carriages for the lamps, screens, etc. Such a track

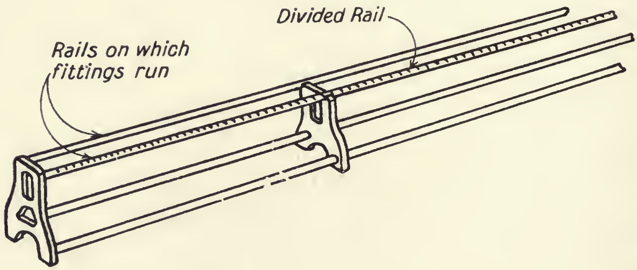


Fig. 54.

should be straight, level, and firmly supported. The front circular rail of the track should have a scale of equal divisions on it to permit distances apart of the various fittings to be read. The length of the track should be from 10 to 15 ft. long.

With such an apparatus more satisfactory photometric measurements may be made.

Experiment I.—Using this bench and the Lummer-Brodhun photometer, the candle-power of an electric lamp should be obtained. For this purpose it is well to mount the lamp on a suitable fitting (as shown in Fig. 55), in order that the lamp may be rotated and the candle-power measured for various positions of the lamp, from which a "light-flux" diagram can be plotted. It is of the greatest importance that when using electric sources of light in photometric work the state of current passing should be known; to this end, therefore, either

an " ammeter " should be put in series in the circuit, or a voltmeter across the lamp terminals.

The candle-power of a lamp determined in this way

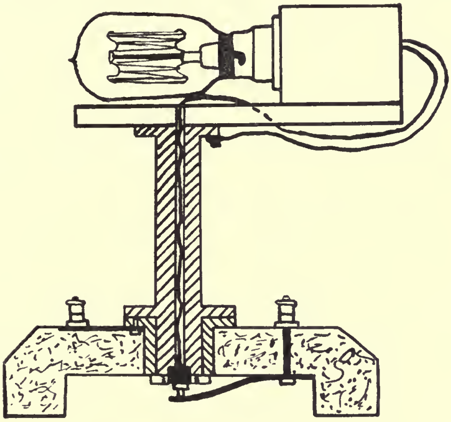


FIG. 55.

would be considered as that measured from the centre of rotation of the lamp serving as a reference point.

Experiment II.—As a practical application of photometry, the following experiment may be performed. It consists in measuring the loss of light in a telescopic instrument.

(*d*) For this purpose a collimator should be used as an accessory to the photometer bench in order to produce

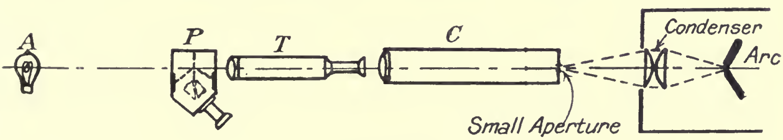


FIG. 56.

a parallel beam of light for passing through the telescopic system under test. The general arrangement of the apparatus for the experiment is shown in Fig. 56.

First, receive the parallel beam from the collimator C into the photometer P on one side (*i.e.* with*out* the telescope in position), and light from an auxiliary lamp A on the other side, and adjust the position of the photo-

meter until a " balance " is obtained. Take the distance d_1 from the photometer to the auxiliary lamp. Focus the telescopic instrument supplied for " infinity," and support it in the position T on the bench so that it receives the parallel beam into the eyepiece of the instrument. Now, the ratio of the intensity of the emergent beam to that of the incident beam should be $\frac{1}{M^2}$ (neglecting internal losses), where M is the magnification of the instrument; the student should prove this for himself. The magnification should be found as explained in Chapter VIII. With the instrument now in position the position of the photometer should again be adjusted until a second " balance " is obtained; let the second distance between photometer and auxiliary lamp be d_2.

Then,

$$\frac{\text{Intensity of final beam}}{\text{Intensity of original beam}} = \frac{(d_1)^2}{(d_2)^2}.$$

By the theory above, if instrumental losses were non-existent, we should find

$$\frac{(d_1)^2}{(d_2)^2} = \frac{1}{M^2},$$

but in practice we shall have

$$\frac{(d_1)^2}{(d_2)^2} = \frac{K}{M^2},$$

where K is the transmission coefficient of the instrument.

The above description gives the outline of the experiment; the student should suggest and carry out all necessary experimental precautions, such as the determination of the current in the lamps, repetition of readings, etc.

NUTTING PHOTOMETER

This instrument is made as an attachment to ordinary spectroscopes for spectro-photometric work. It is used for the comparison of light sources as to their intensity

of radiation for the various wave-lengths of the spectrum; it may also be used for absorption work. The optical system of the instrument is shown in Fig. 57. Light from the two sources are admitted through the apertures A_1 and A_2. (For absorption work it is better to use a "split" beam from one source.) Light through A_2 passes through a stationary Nicol (or Glan Thompson) prism N_1; that through the aperture A_1 is brought in

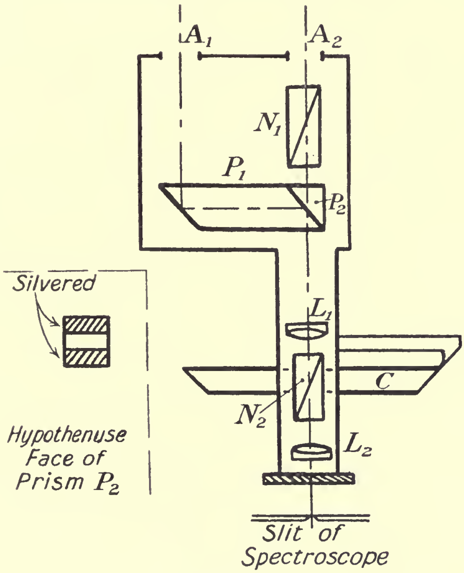


FIG. 57.

the direction indicated by means of the prism P_1. The inner surface of the prism P_2 (which is balsamed to P_1) is silvered with two horizontal strips, so that light from A_1 will be reflected along the path A_2L_1, and the light from A_2 will pass straight through the unsilvered strip. The lens L_1 renders the light parallel before passing through the rotating Nicol N_2. The lens L_2 focusses an image of the "tri-partite" field on the slit of the spectroscope. The rotation of the Nicol N_2 can be measured by a divided circle C and pointer. The appearance in

the eyepiece of the spectroscope should be that of three sharply defined spectrum "bands," the centre one of which is varied in intensity by rotation of the Nicol N_2. The source or specimen (if for absorption) to be tested is placed in front of the aperture A_1, and the intensity of the centre band varied, until a "match" with the outer bands is obtained. Since only one of the incident beams is polarized, the intensity of the light varies as the square of the *cosine* of the angular turn of the Nicol N_2.

Experiment.—With the Nutting photometer and spectroscope, the intensities of illumination for various parts of the spectrum, of, say, a piece of cobalt glass or some solution, may be determined, and a graph plotted showing the relationship between intensity and wave-length.

LUMMER-BRODHUN SECTOR

This piece of apparatus is frequently used in connection with photometric measurements. It serves as a means of varying the intensity of a beam of light by a known amount, by inserting a revolving sector (whose apertures are adjustable) in the path of the said beam. Fig. 58

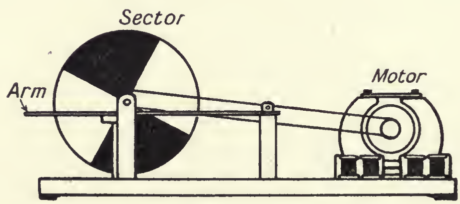


FIG. 58.

illustrates the apparatus, and, as will be seen, an electric motor is employed for driving purposes, whilst an adjustable arm will be noticed for varying the aperture of the sector whilst in motion.

When in use the speed of the sector should be arranged, so that on looking into the instrument with which the sector is being used no flicker of the field is noticeable.

Under this condition, the intensity of the light transmitted by the sector may be taken as being proportional to the angular aperture.

The instrument may be used on the photometer bench in the path of one of the beams of light, and serves as a means of varying its intensity without the necessity of moving the source of light relative to the photometer.

CHAPTER IV

SPECTROMETER MEASUREMENTS

(*a*) THE SPECTROMETER

THE spectrometer is an instrument of fundamental importance for the measurement of refractive index (see Chapter I., section (*c*)). The essential parts of the instrument comprise a "collimator" SL_1 (Fig. 59), a rotating prism table T, and a telescope L_2E on a movable arm. The collimator consists of a metal tube, at one end of which is an achromatic lens L_1 and at the other

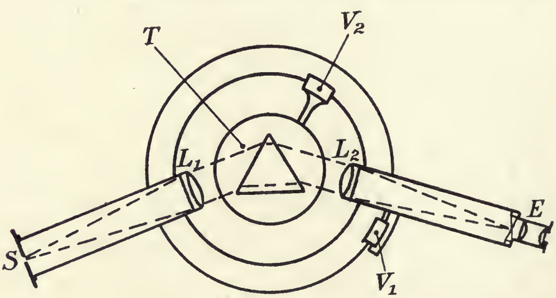


FIG. 59.

end a vertical "slit" S in the focal plane of the lens. Light diverging from this slit is rendered parallel (or collimated) by L_1 and "parallel light" falls on the prism. The light having passed through the prism, the spectrum thus produced is brought to a focus by means of the lens L_2 of the telescope in its focal plane, and this image is viewed by an eyepiece E. The telescope rotates so that it is always directed towards the axis of rotation of the prism table, and is provided with a vernier V_1, which moves over a divided circle concentric with the prism

table. To the latter there is also a vernier V_2 attached, which moves over the inner edge of the dividing of the circle. It is necessary that the instrument should be thoroughly rigid, and precision must be exercised in the fitting of the bearings, verniers, and circle. It will be found less expensive if such an instrument is bought outright rather than to try and construct such an instrument in one's workshop. A selection of numerous "makers" will be found in the "Dictionary of British Scientific Instruments."

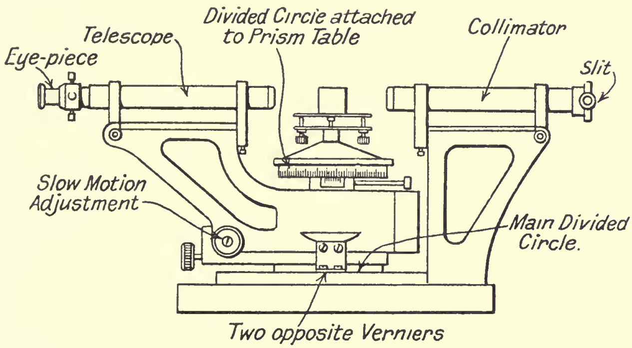


FIG. 60.

Fig. 60 shows a convenient type of spectrometer for laboratory use (by Watts).

Adjustments.—The following adjustments are necessary before commencing an experiment with the spectrometer:

1. *To adjust the eyepiece.*—The eyepiece lens system is movable in the tube which carries the cross-webs. Hold a piece of white paper in front of the telescope objective so as to reflect light into the telescope, then move the eyepiece in or out until the cross-lines are sharply defined.

2. *To adjust the telescope.*—Direct the telescope towards some distant object, such as a church spire, and move the tube carrying the eyepiece and cross-wires (usually by a rack-motion) until the image of the distant object is seen sharply defined at the same time as

the cross-lines. To be sure of this see that there is no parallax between the two. The telescope is now in normal adjustment.

3. *To adjust the collimator.*—Illuminate the "slit" of the collimator. Swing the telescope into such a position so that it and the collimator tube are in the same straight line; and then, while looking through the telescope, move the slit in or out until there is no parallax between its image and the cross-lines. Set the slit vertical.

4. There are two alternative methods of focussing the collimator for parallel rays which should be taken note of. First, Schuster's method: the prism is set in the position of minimum deviation, and the telescope turned so that the image of the D line or some other convenient line is seen. The telescope is then turned a little to one side of the image; it is evident that there are now two positions of the image, one on each side of that of minimum deviation, which will bring the image of the line again into the centre of the field of view of the telescope. The prism is turned to these two positions in succession, and the line observed in each case; if the line appears in perfectly good focus at each time, then the telescope and collimator are both adjusted for parallel rays. If, as is more probable, the focus of the line appears better at one position than at the other, the following procedure should be adopted. The prism is first turned to one position, and then the collimator is focussed until the line is seen perfectly sharp; after turning the prism to the other position the telescope is focussed until the line is again sharp. After one or two repetitions it will be found that the condition will be obtained so that the line remains in perfect focus whichever way the prism is turned. When this is obtained, telescope and collimator will be in adjustment for parallel light.

The second method is by Lippmann, who employs two strips of "plane parallel" glass, which are set one above

the other and at right angles to one another (see Fig. 61). This apparatus is set in the path of the beam from the collimator; if these rays be truly parallel no effect will be

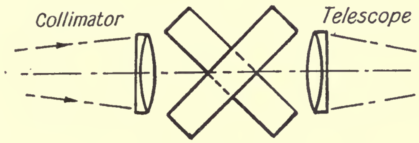


Fig. 61.

produced, but if they are convergent or divergent, the upper and lower halves of the image of the slit will appear relatively displaced.

(b) MEASUREMENT OF PRISM ANGLES

First Method.—Let ABC (Fig. 62) be a prism, of which the angle A is required to be measured. The prism is placed on its table and levelled so that the faces AC and AB are vertical. Adjust its position and that of the telescope so that an image of the slit is formed on the cross-wires by means of light reflected from the face AB. Without moving the telescope, rotate the prism table until the face AC acquires such a position that light reflected from this face forms an image of the slit on the cross-wires. In order for this to be so, it is obvious that AC must take up a position parallel to that previously occupied by AB, which is equivalent to rotating the prism through the angle CAD. Therefore the angle of the prism A is the supplement of this angle, which equals $180° - \angle CAD$. The angle CAD is obtained by the readings taken from the divided circle for the two positions of the prism table.

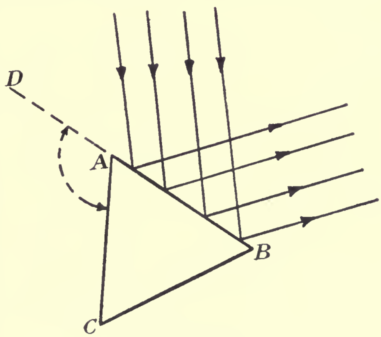


Fig. 62.

Second Method.—Arrange the prism with angle to be measured towards the axis of the collimator so that the parallel beam from this falls partly on the face AB (see Fig. 63) and partly on AC. Move the telescope round until an image of the slit is seen by light reflected from one of these surfaces. Set this image on the intersection of the cross-wires and take the reading of the telescope position from the divided circle. Leaving the prism table stationary, move the telescope round until the image of the slit is again seen, but this time after reflection from the second surface, and take a second reading from the circle. The difference between these two readings is equal to twice the angle BAC.

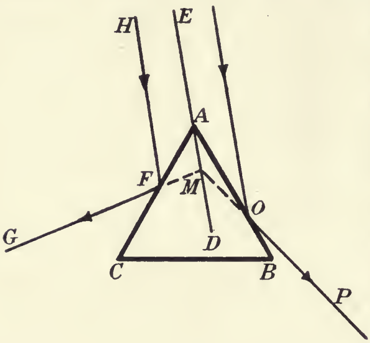


FIG. 63.

Proof :

Produce EA to D, GF to M, and PO to M.

Then, because HF and ED are parallel

$$\angle HFA = \angle FAM.$$

And $\angle GFC = \angle AFM$ (vert. opposite angles).

But $\angle HFA = \angle GFC$ (by reflection).

$$\therefore \angle FAM = \angle AFM.$$

$\angle GMD = \angle FAM + \angle AFM$ (two interior and opposite angles).

$$\therefore \angle GMD = 2\angle FAM.$$

Similarly, $\angle PMD = 2\angle OAM$.

So that $\angle GMP$ (the angle moved through by the telescope) $= 2$ BAC.

Experiment.—Measure the angles of the prism supplied to you by the two methods described. Test the accuracy of your results by seeing if the sum of the three angles added together equal 180°.

(c) MEASUREMENT OF REFRACTIVE INDEX AND DISPERSION

With any prism there is an important relation between the Refractive Index (n), the Vertical Angle of the prism (A), and the Angle of Minimum Deviation (D) (see Chapter I.). The equation connecting these three quantities is written as :

$$n = \frac{\sin \frac{(A+D)}{2}}{\sin \frac{A}{2}}.$$

Before going any further it is well to look at the proof of this formula.

Let the angle BAC (Fig. 64) be the measured vertical

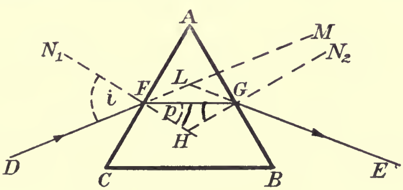


Fig. 64.

angle of the prism. Call this angle "A," and suppose the prism to be in the position of "minimum deviation" (see Chapter I.), *i.e.* $\angle EGN_2 = \angle DFN_1 = i$, and $\angle HGF = \angle HFG = r$.

Now, in the figure AFHG the angles AFH and AGH are right angles, so that the angles FAG and FHG must together equal two right angles, from which we see that :

$\angle FAG = \angle HFG + \angle HGF = 2\angle HFG = 2r$;

i.e. $\angle BAC = 2r$ or $r = \frac{A}{2}$.

Also $i = \angle DFN_1 = \angle HFL$.

But $\angle HFL = \angle GFL + \angle GFH$, which equals

$$\frac{\angle MLE}{2} + r \text{ or } \frac{D}{2} + \frac{A}{2}.$$

So that $n=\frac{\sin\ i}{\sin\ r}=\frac{\sin\frac{(A+D)}{2}}{\sin\ \frac{A}{2}}$.

Making use of this formula the refractive index of the prism may now be determined. First of all, however, it must be remembered that a prism produces a spectrum and that the various colours or wave-lengths are deviated by different amounts on passing through the glass, the red being the least refrangible and the violet the most refrangible. Therefore "n" will vary, depending on the wave-length of the light; thus it is that certain definite wave-lengths in the spectrum have to be decided on in order that some standard of comparison may be formed for identification of all glasses, also liquids. These wave-lengths are :

Wave-length.	Produced by	Notation.
·00005893 cms.	Sodium Flame	D line
6563 ,,	Hydrogen Tube	C ,,
4861 ,,	,, ,,	F ,,
4102 ,,	,, ,,	G_1 ,,

Experiment.—Place a sodium flame * in front of the slit of the spectrometer—open the slit fairly wide. Put the prism on its table and observe the image of the slit with the telescope after the light has been refracted through the prism. If the telescope and collimator have been carefully set for parallel light previously, the slit-image should be well defined ; close the slit down gradually until a very narrow image is obtained, and if necessary focus it sharply by means of the rack motion of the telescope.

To set the prism at minimum deviation.—Rotate the

* A very suitable "sodium flame" may be made by employing a "Meker Burner," as supplied by Messrs Baird & Tatlock, Hatton Garden, and by placing common salt on the "grid" at the top of the burner.

prism table and observe the movement of the image; if it goes outside the field of view of the telescope, move the latter round the circle so as to keep it in view; but on continuing to move the prism table in the same direction the image will reach a limiting position and then commence moving in the opposite direction. When the image reaches this position set the intersection of the cross-wires on it; this is the position of minimum deviation for the sodium line. (If two sodium lines are seen, set the cross-wires midway between the two, for there are actually two lines with six Angström units between them.) The reading from the circle should be

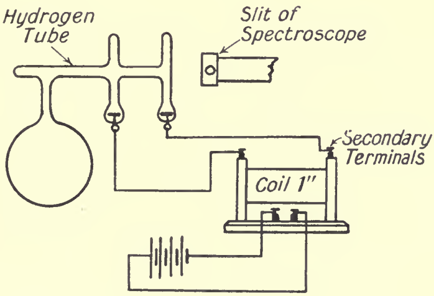


FIG. 65.

taken, then remove the prism and take the reading when the telescope is set for the slit image as viewed directly, *i.e.* in the same straight line as the collimator. The difference in these two readings will give the " deviation " D. The angle of the prism A has already been determined, so that the refractive index for sodium light (denoted n_D) can be calculated.

Similarly, by placing a hydrogen tube * in front of the slit, the refractive indices of the prism for the C (red), F (green), and G_1 (blue) lines may be obtained.

Dispersion is denoted by the difference in refractive

* A very convenient type of " Hydrogen Tube " for this work is the " Guild " form. This is shown in Fig. 65, with the electrical connections when accumulators and " coil " are employed for discharging.

index between the two wave-lengths in question, and is written usually:

C to D = ·00481 (for instance)
D to F = ·00970
F to G_1 = ·01741

Dispersive Power of the prism is given by the formula $D = \frac{n_{G_1} - n_C}{n_D - 1}$, where n_{G_1} and n_C are the values for the refractive index for the G_1 and C lines respectively.

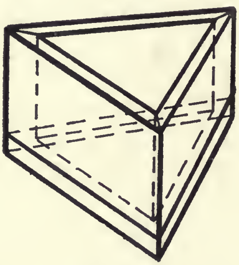


FIG. 66.

Refractive Index of Liquids.—The refractive index and dispersive power of liquids may be found by the above methods by using a hollow glass prism with the sides of "plane parallel" and optically flat glass, and filling the prism with the liquid under test. Such a prism is shown in Fig. 66. Plaster of Paris makes a good cement to secure the sides and base.

(*d*) **REFRACTIVE INDEX BY IMMERSION.** (See *Trans. Opt. Soc.*, vol. xvii., No. 3, Dec. 1916. Mr L. C. Martin on "Refractometry and Identification of Glass Specimens.")

A very useful means of obtaining the refractive index of specimens of glass in a rough or unpolished state, or of lenses, is by immersing the specimen to be tested in a liquid of the same refractive index contained in one of the hollow prisms shown in Fig. 66. The whole can then be mounted on the spectroscope and the usual necessary measurements taken.

For this purpose, however, it is necessary to have a liquid of variable refractive index. Carbon disulphide and alcohol mixed together provide a readily adjustable solution; in practice it is found best to start with pure carbon disulphide in the prism, immerse the specimen, and then dilute the solution with the alcohol. The strength of the liquid should be adjusted so that its index is very slightly higher than the value required to focus

sharply (on looking through the telescope) any particular line of the spectrum (*e.g.* the sodium lines), and the evaporation of the carbon disulphide, which usually occurs faster than the alcohol, will presently bring the line into focus very slowly and distinctly. At the moment of sharpest focus, the angle of "minimum deviation" is taken in the usual way, and the refractive index worked out in the usual way from the formula.

One of the most important factors of the whole experiment is that the liquid in the prism should be kept homogeneous. To this end it is necessary to have the liquid mechanically stirred; a small "propeller blade" driven at a suitable speed in the liquid by a small electric motor will secure this condition. The motor must not be mounted on the same table as the spectroscope, as the vibration will interfere with the readings. The method is suitable for any rough small pieces of glass, except forms approximating to plane parallel plates.

Experiment.—Find the refractive index of the specimen supplied to you by the above method for "D" light (λ 5893), and then for the C, F and G_1 lines (hydrogen); also determine the dispersion and dispersive power.

(*e*) DETERMINATION OF THE WAVE-LENGTH OF LIGHT BY MEANS OF A DIFFRACTION GRATING

This experiment again involves measurements with the spectroscope, but instead of using it for the determination of refractive index it is to be used for finding the wave-length of certain lines in the spectrum. For this purpose a "Grating" is employed; this consists of a piece of speculum metal which has its surface ruled with a great number of parallel lines very close together (the rulings are about 14,000 lines per inch). A very suitable transmission grating is made by taking a "cast" in gelatine from a metal grating and floating it on a piece of parallel glass.

The full theory of the grating must be revised from textbooks (Edser's "Light for Students" or Baly's "Spectro-

scopy"), and cannot be dealt with in this book, but it will be sufficient to say here that the spectrum produced by a grating is due to the "interference" of waves passing through the spaces in the grating. Let (Fig. 67) AB and CD be two adjacent apertures in the grating,

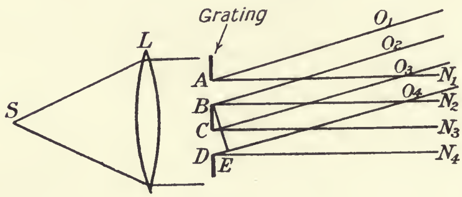


Fig. 67.

and that parallel light is incident in the direction indicated by the arrow, *e.g.* from a collimator SL, S being a slit parallel to the apertures of the grating. Now the supposed "ether particles" lying in the apertures AB and CD become sources of vibration which proceed chiefly in the direction towards N_1, N_2, N_3 and N_4, but also, however, in other directions, as towards O_1, O_2, O_3 and O_4. If the former rays are brought to a focus by means of a lens (*i.e.* the telescope objective), they will produce a bright image of the slit without any mutual interference taking place, whereas the case with the diffracted rays O_1, etc., is rather different. In order to investigate the "interference" among the latter, the straight line BE is drawn perpendicular to DO_4, when the line DE will represent the difference in path travelled by the two outside rays DO_4 and BO_2 and also between the two outside rays CO_3 and AO_1, and, therefore, also the difference in path travelled by every pair of corresponding rays in the two "pencils." If now DE is equal to any odd number of half wave-lengths, it follows that for every ray in one pencil there is a corresponding ray in the other "pencil" at opposite "phase," and, therefore, total interference takes place when the rays are combined at the focus of the lens. The same holds good for every adjacent pair of apertures of the grating.

But if DE be equal to any even number of half wave-lengths, then every corresponding ray in the two pencils will be at equal "phase," and, therefore, the rays from these two apertures and every adjacent pair will combine at the focus of the lens to give a bright image of the slit. Thus it will be seen that from a "grating" a spectrum will be formed on "either" side of the direct image of the slit, and the deviation of lines in the spectrum from the direct image is dependent on the wave-length, *i.e.* the length DE decides the angle DBE which equals $\angle N_4DO_4$. This gives a means of determining experimentally the wave-length of any particular line in the spectrum, for the deviation N_4DO_4 can be measured with the spectroscope, and the distance DB can be obtained from knowledge of the number of lines per inch of the rulings:

$$\therefore \frac{DE}{DB} = \sin \angle DBE.$$

If DE equals one wave-length the spectrum seen in the direction O_4 is known as the "first order spectrum." If DE equals "two" wave-lengths a "second order spectrum" will be seen, and so on.

Experiment.—Perform all necessary adjustments to the spectroscope, and then, illuminating the slit, take the reading of the telescope when the image of the slit is on the cross-wires as seen "directly," *i.e.* in the same straight line. Set up the "grating" in a vertical position over the centre of the prism table. It is important, first of all, that the grating is set "normal" to the axis of the collimator. To do this, move the telescope round the circle until it is exactly 90° from its previous reading, rotate the prism table with the grating on it until light from the collimator is reflected off the plane glass surface of the grating into the telescope and an image of the slit is made to coincide with the intersection of the cross-wires. The grating will then be at 45° to either telescope or collimator. Take the vernier readings of the prism table and then rotate it to a position 45° from its previous

reading. The grating will then be at right angles to the axis of the collimator; the plane glass side of the grating should be towards the O.G. of the collimator.

For this experiment a very good source of light to use is a mercury vapour lamp, as it has a few prominent and well-spaced lines in its spectrum. These lamps can be obtained from the Cooper, Hewitt Co., and are very suitable for the laboratory. However, if this is not available, the sodium flame and hydrogen tube may be used as before.

Direct the collimator towards the source, and on moving the telescope to about 18° from the direct reading, the spectrum (first order) will be seen in the field of view. Set the cross-wires on some definite line (if the mercury spectrum is used two yellow, one bright green, and one violet line will be seen), and take the reading of the telescope verniers, take also a reading when the telescope is on the other side of the "direct" position; these two values should be the same, of course. Calculate the wave-length of the line from the formula:

$$\lambda = d \sin \theta,$$

where λ = the wave-length,

d = the mean distance apart of the rulings,

and θ = the angle between the direct and diffracted image of the particular line in question.

Repeat the experiment for the other lines in the spectrum, then move the telescope still further round, when the spectrum will be seen to repeat itself, this being the "second order." Take readings for the same lines in this spectrum and again determine their wave-lengths; in this case, from what has been said before, the formula will be:

$$\lambda = \frac{1}{2} d \sin \theta,$$

and $\lambda = \frac{1}{3} d \sin \theta$ for the third order.

Tabulate all your results.

Table of Prominent Lines in the Sodium, Hydrogen, and Mercury Spectra

Line.	Wave-length in cms.	
D_1 Sodium	·00005890	orange
D_2 ,,	5896	orange
C Hydrogen	6563	red
F ,,	4861	blue
G_1 ,,	4102	violet
Mercury lines	5791	yellow
,, ,,	5769	yellow
,, ,,	5461	green
,, ,,	4359	violet

(*f*) **CALIBRATION OF THE SPECTRUM**

The use of the spectrum for the purpose of analysis is now well known, gases and metallic substances each having

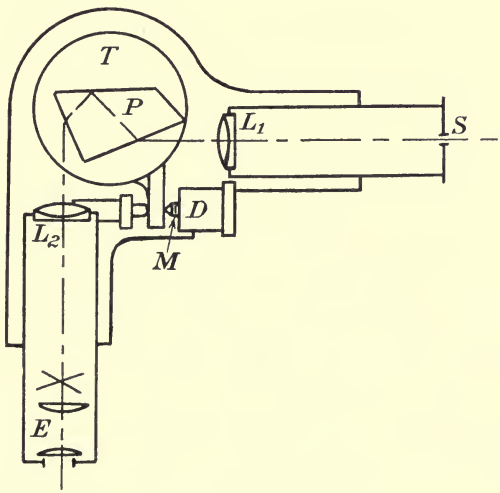


FIG. 68.

a characteristic spectrum when seen through the spectroscope. Thus it is that the spectrum may be "mapped out"

by simply measuring the various deviations (with one of the previously described spectroscopes) for certain lines of the spectrum of known wave-length, and plotting a curve one against the other. By the aid of this curve we can find the wave-length of any unknown line.

Experiment.—Determine the values and draw out such a curve.

There is a certain type of spectroscope, however, which gives the wave-length of any spectrum line direct, without the necessity of having to make a calibration curve. It is known as the Constant Deviation Spectrometer, and employs a prism of the type shown in the last section of Chapter I. A plan of the instrument is shown in Fig. 68. SL_1 is a collimator and EL_2 a telescope set accurately at 90° to one another. P is the "constant deviation" prism through which the light from the collimator travels as indicated in the figure, and becomes dispersed. This prism rests on a circular table T which is rotated by means of a micrometer screw M; to this screw is attached the drumhead D, which is engraved in wave-lengths. To use the instrument, all that is necessary is to set the drum to read a known wave-length, such as $\lambda 5890$, then move the prism on its table by hand until that particular line comes coincident with the intersection of the cross-wires in the telescope. Clamping the prism in this position, the instrument is now adjusted. By bringing any other line of the spectrum on to the cross-wires, its wave-length may be read off direct from the drum, the calibration of which has been carried out once for all by the makers.

CHAPTER V

DETERMINATION OF RADII OF CURVATURE OF SURFACES

(*a*) THE most usual instrument that is employed for determining the radius of curvature of lens surfaces is a "spherometer." There are numbers of types of this instrument—for example: (i) the "three-legged," (ii) the "ring" type, (iii) the "Aldis" type, (iv) Abbé type, etc., but all spherometers are dependent on a certain formula.

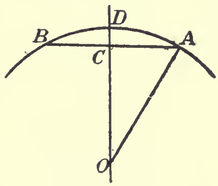


Fig. 69.

This formula is deduced in the following manner (see Fig. 69):

Let ADB be part of the circumference of a spherical surface (*e.g.* a lens) in section, and we require the radius OA or OD of the surface. Draw AB perpendicular to OD. Then the $\triangle$ OAC is a right-angled triangle, and therefore $OA^2 = OC^2 + AC^2$; also $OC = OD - CD$.

Call $OA = R$, $CA = r$, and $CD = h$.

Then $R^2 = (R - h)^2 + r^2$.

Hence $2Rh = r^2 + h^2$

and $$R = \frac{r^2 + h^2}{2h}.$$

Now, the spherometer is a means of obtaining the distances $CD = h$ and $CA = r$, from which R (the required radius of curvature) is calculated.

Fig. 70 shows a three-legged spherometer, and, as will be seen, it consists of a small tripod, in the centre of which is mounted a very finely pitched micrometer screw with a divided disc attached to it. This divided disc reads intermediate values of divisions of the vertical scale

shown in the figure on one of the legs. When using the instrument, it must first be placed on a flat surface (such as an optical "flat"), and the micrometer screw moved

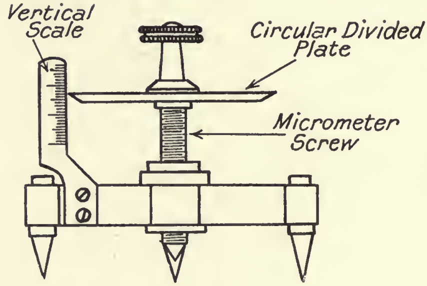


FIG. 70.

up or down until all four "feet" are exactly in contact with the surface at the same moment; the reading on the scale and divided disc should then be taken; this should be the zero of the instrument. After this, place the instrument on the surface whose radius is required, and again move the micrometer screw up or down according as the surface is convex or concave and take a second reading. (All readings should be a mean value of a number of settings.) The difference between the readings taken on the flat and on the curved surface will give the value "h" in the formula. The value "r," which is the radius of the circle on which the three legs lie, is very often engraved on the instrument; however, for accurate measurements this should always be checked by measurement with a travelling microscope. This may be done in two ways *: either by measuring the distance between

* *Proof.*—In Fig. 71, A, B, and C are the three "feet" of the spherometer in plan, and the distance AB, BC, and CA are measured. Call their mean value "p."

Then—

$$r=\frac{2}{3}p \cdot \cos 30^\circ$$

$$=\frac{2}{3}p \cdot \frac{\sqrt{3}}{2};$$

$$\therefore r=\frac{p}{\sqrt{3}}.$$

This method for obtaining "r" is more especially useful when the "feet" of the spherometer are worn flat.

the centre leg and each outside leg in turn, and taking the mean value, or by obtaining the mean distance between each outside leg and dividing by $\sqrt{3}$.

Experiment.—Determine the radius of curvature of the convex and concave surfaces supplied to you with the three-legged spherometer. Check the value for "r" by means of a measuring microscope.

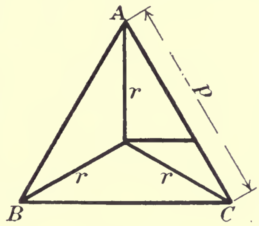


Fig. 71.

"*Ring*" *Type Spherometer.*—This type of spherometer is very similar to the three-legged, but it involves the use of a metal "ring" in place of the three legs. The micrometer screw and drumhead are used in a similar manner, but to determine the value "r" in the previously mentioned formula the maximum internal diameter of the ring must be measured for convex surfaces and the maximum external diameter for concave surfaces. The instrument is shown in Fig. 72.

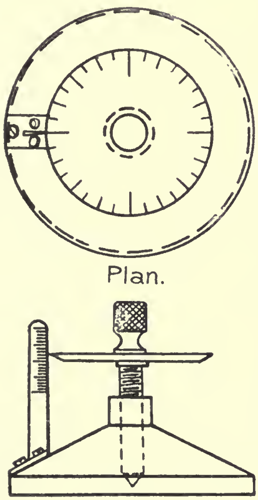


Fig. 72.

Abbé Type.—A rather better and more accurate type of spherometer is the Abbé type. The instrument is shown in Fig. 73. It uses a well-fitting steel plunger sliding up and down in a vertical direction. The surface to be tested is placed on an accurately turned ring situated at the top of the instrument, whilst the spherical nose of the plunger is kept in contact with the surface by means of a counterweight suspended over small pulleys. Attached to the plunger is an engraved scale divided in tenths of a millimetre (·1 mms.), which is observed by a microscope with micrometer eyepiece, and readings may be taken to $\frac{1}{1000}$th of a millimetre. As in the last case,

the internal and external diameters of the particular ring in use must be measured. A series of rings of various

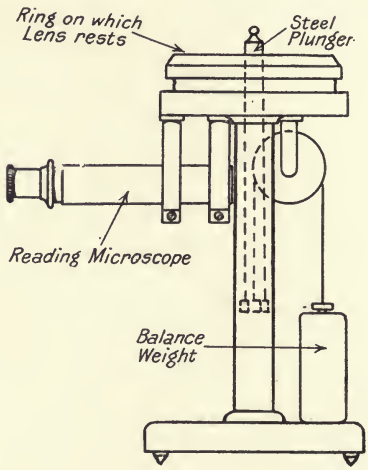


FIG. 73.

diameters is supplied with the instrument for use with corresponding sizes of lenses.

Aldis Type.—A still better and probably the most accurate instrument of its kind is the "Aldis" Spherometer (an illustration of it is given in Fig. 74). The surface to be tested is allowed to rest on three small spheres, and the micrometer screw is screwed up to touch the surface. Opposing the screw is a weighted plunger which rests on the other side of the lens; by this means the instrument is rendered extremely sensitive, for contact between the point of the micrometer screw and the surface is at once detected by touching the edge of the lens with the finger-tips and judging the ease of rotation. If the lens revolves freely the micrometer screw is too high, and if the lens will not revolve the screw is not touching the surface; a position will be found when the lens will just and only just revolve, this will be when the point of the screw is in correct contact. The drum attached to the micrometer screw is 2 in. in diameter, and readings

may be taken to ·00001 of an inch. In using the spherometer formula with this instrument the value "R" is the radius of curvature of the surface + the mean radius of the spheres; therefore on arriving at the calculated

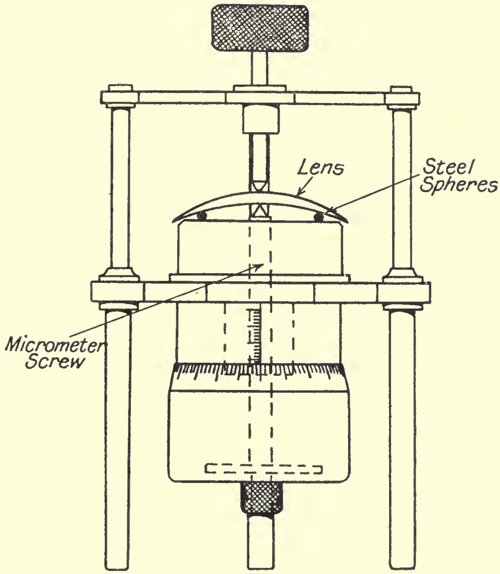


Fig. 74.

value of "R," to obtain the true radius of curvature of the surface the radius of the spheres must be subtracted. (If a sketch is drawn this will become evident; it is equivalent to working on a sphere of radius $R+x$, where x is the radius of the spheres.)

(b) CURVATURE OF "SMALL DIAMETER" SURFACES

It is obvious that the use of the spherometer is limited by the diameters of lenses; when lenses are from 1 in. in diameter downwards, and more especially microscope objective lenses, some other method than the spherometer has to be employed. The following method gives a good and very accurate way of determining the radii of curvature of small diameter lens surfaces.

Fig. 75 gives a diagrammatic explanation of the method. Light from a distant lamp is reflected into the eyepiece of a microscope by means of a plane glass reflector G. (For this experiment it is better to remove the field lens of the eyepiece.) Then an image of the lamp will be formed in the focal plane of the eye lens at I_1, and also a second image by the micro. objective O at I_2. Now, if the surface to be tested is placed at I_2, light will be reflected from it, and, returning along its original path, will form another image at I_1, so that an eye placed at E will see this image; the first image will, of course, not be seen, as the light is travelling in the wrong direction. Similarly, by placing the surface in a second position, as at I_3 (when all the rays from O strike the surface *normally*), another image will again be seen at I_1.

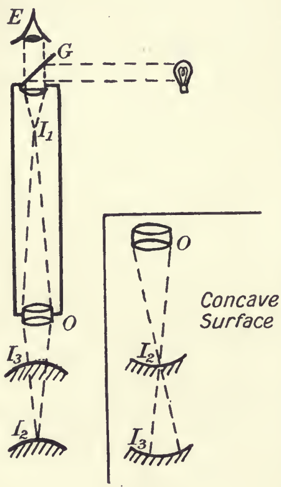


FIG. 75.

The distance between these two positions of the surface, namely, at I_2 and I_3, will give the radius of curvature of the surface. Refer to method of determining the curvature of a convex mirror (Chapter II., section (*c*)).

For measuring this distance I_2I_3 accurately, either the microscope must remain fixed and the lens move in a vertical direction, or the lens remain stationary and the microscope move on a vertical axis. In the latter case the experiment is simplified by employing a measuring microscope with a special adapter (made by the Cambridge & Paul Scientific Instrument Co., Cambridge), described in Chapter VI., section (*a*), as this instrument can be used very conveniently in a vertical direction, and measurements taken to a thousandth of a millimetre. In the former case a simple piece of apparatus may be made up by adapting a Brown & Sharpe micrometer

head to the stage of a student's microscope, as depicted in Fig. 76; in this case the lens would be attached to the movable head and moved up and down with it, readings being taken from the micrometer drum for the two positions of the lens surface I_2 and I_3 in Fig. 65.

Experiment. — Determine the radius of curvature of the convex and concave surfaces supplied to you by one of the above methods. (The method applies equally well to concave surfaces as well as convex.)

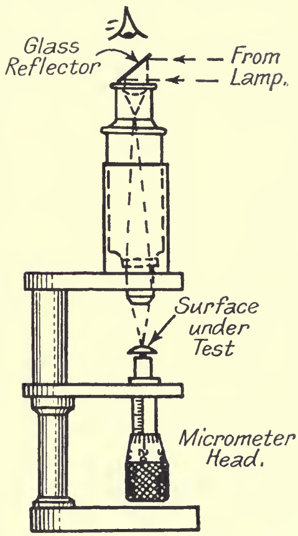


Fig. 76.

(*c*) CURVATURE : NEWTON'S RINGS METHOD

Thoroughly clean a long focus convex lens and a piece of plate glass (flat), press them together and examine the reflection of the sky near the point of contact. A dark spot surrounded by a series of coloured rings will be seen. By using monochromatic light, such as a sodium flame or mercury vapour lamp, many more rings, alternately light and dark, may be seen. It will be found that the rings are closer together as they are larger, also it will be noticed that the rings are closer for yellow than for red light, and still closer for green or blue light. Their formation is due to the interference between the light reflected from the front and back surfaces of the air film between the lens and glass plate. The rings may also be seen by transmitted light; in this case, however, they are much fainter.

Let us consider the theory; and to simplify this it is better if we are concerned first with the rings seen by *transmitted light.* Let OAB be the lower surface of the lens resting on a plane surface OMN (Fig. 77). In the figure, O is the point of contact of the lens and surface, so

the complete figure is symmetrical about the point O. Consider light coming in the direction LO normal to the surface OMN. At a given point, A, the thickness of the air film between the two surfaces is AM. Part of the light incident at A passes straight through the film at this point without reflection; another part is reflected at M,

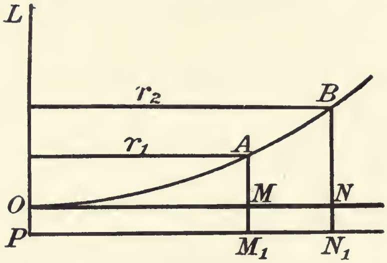


FIG. 77.

and again at A, and finally passes out at M in the direction MM_1. It therefore suffers a retardation in path equal to 2AM. If 2AM is equal to half a wave-length of the light considered, or any multiple of half a wave-length, the two portions of light differ in phase by half a period and "interfere," producing a dark band at A. If, however, $2AM = $ a whole wave-length (λ) or any multiple of λ, the two portions of light combine at M in the same phase, and A is the middle point of a bright band. At O, where there is no difference of phase, there is a bright spot. On passing outward from O the thickness of the air film increases until it becomes equal to $\frac{\lambda}{4}$. At this point there is a dark ring: still further out the thickness has increased to $\frac{\lambda}{2}$, and at this point there is a bright ring: then when the thickness is $\frac{3\lambda}{4}$ there is a second dark ring: and so on.

If M is the position of a dark ring and R is the radius of curvature of the surface OAB, then by a property of the circle

$$2R \times AM = r_1^2 \text{ nearly};$$

or
$$2AM = \frac{r_1^2}{R}.$$

If B is the position of the next dark ring, $2BN = \frac{r_2^2}{R}$. Hence $\frac{r^2}{R}$ must equal $\frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}$, etc., or generally $r^2 = \frac{(2n+1)\lambda}{2} R$, where "$n$" is any integral number or zero.

The radii of successive dark rings, therefore, increases as the square roots of the odd natural numbers, and the areas of the annuli between successive rings are the same.

Also
$$2BN - 2AM = \lambda:$$

therefore
$$\lambda = \frac{1}{R}(r_2^2 - r_1^2).$$

If B is not the next but the n^{th} dark ring after A, we have

$$n\lambda = \frac{1}{R}(r^2_{(n+1)} - r_1^2).$$

In this experiment the wave-length of the light being used would be known (that of sodium light being ·0000589 cms. or that of a mercury vapour lamp passing through a green filter being ·0000546 cms.), so that the expression may be made, with a knowledge of the radii of the rings, to give the value of R.

When the rings are viewed by *reflected* light dark bands are seen when the retardation within the film is λ or any multiple of λ.

This is due to the two reflections not taking place under the same conditions. In the transmitted light both reflections are from surfaces of the glass, but for reflected light one reflection (at A) is from a surface of air and one (at M) from a surface of glass. From this cause there is produced a retardation of phase $\frac{\lambda}{2}$ which must be added to that due to the difference in paths.

Experiment.—With the lens and glass "flat" supplied to you, form the rings by placing the two in contact, the flat surface resting on the curved one (see Fig. 78). The

system may be made quite stable by small pieces of soft wax at A and B.

The measurement of the rings formed by *reflected* light is effected by means of a measuring microscope. The Cambridge & Paul Scientific Instrument Co.'s type, as described in Chapter VI., section (a), is very suitable.

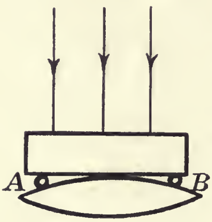


Fig. 78.

The point of contact of the two surfaces is viewed with the microscope, and is illuminated by means of a "vertical illuminator" in the microscope. This piece of apparatus is shown in Fig. 79, and consists of a small plane glass plate placed diagonally between the objective and the microscope body tube ; in this way light from the monochromatic source is reflected down normally upon the "flat" and lens. As a monochromatic source, light from a mercury vapour lamp filtered through a green gelatine filter gives best results for this experiment, although, of course, a sodium flame may be used.

By means of the microscope measure the diameter of the 3rd, 5th, 6th, and 7th dark rings—also the 15th, 16th, and 17th—or even of three rings further from the centre, say the 25th, 26th, and 27th if possible.

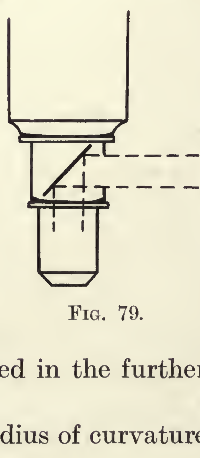


Fig. 79.

Calculate an approximate value of the radius from the 3rd and 7th rings say —correcting this value from calculations made from the radii of the most widely separated pairs measured, say the 5th and 25th, the 6th and 26th, etc.

The determination from the 3rd and 7th rings will prevent mistakes being made if a wrong number of rings is counted in the further work.

In this way, applying the formula, the radius of curvature of the surface R may be obtained.

(*d*) RADII OF CURVATURE: REFLECTION METHOD (KOHLRAUSCH)

This experiment gives another convenient method of determining the radius of curvature of lens and mirror surfaces; moreover, the method is applicable to both large and small surfaces.

Fig. 80 shows the method employed. Two light sources,

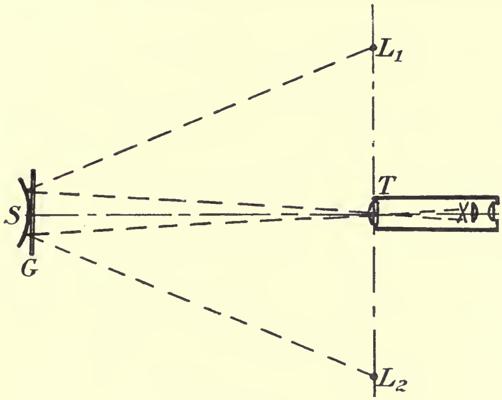


FIG. 80.

such as candle flames, or preferably two illuminated vertical slits, are placed at L_1 and L_2. At the mid-point between these two is situated a telescope T, so that the object-glass lies in the same straight line as the two lights. The surface to be tested, either convex or concave, is placed at S, at a distance not less than 3 metres, so that on looking through the telescope two images of the light sources will be seen by reflection from the surface under test. If, now, a glass scale G is placed in contact with the surface, the separation of the two images may be measured. From this and a knowledge of the distances ST and L_1L_2 (these can be measured with a steel tape), the radius of curvature of the surface may be obtained from the following formula:

$$\left.\begin{aligned} r &= \frac{2dl}{\mathrm{L}-2l} \text{ for a convex surface} \\ \text{and} \qquad r &= \frac{2dl}{\mathrm{L}+2l} \text{ for a concave surface} \end{aligned}\right\};$$

where r = the radius of curvature of the surface,
d = the distance ST,
l = the measured separation of the images on the scale,
L = the distance apart of L_1 and L_2.

The student should prove these formulæ for himself from previous knowledge; however, the proof for a convex surface is given below:

The line L gives an image behind the spherical surface at a distance x, by the rule $\frac{1}{x}=\frac{1}{d}+\frac{2}{r}$ ($\frac{1}{2}r$ is the focal length).

The length y of this image is also given by

$$\frac{y}{L}=\frac{x}{d}.$$

From these two formulæ we find

$$x=\frac{dr}{2d+r} \text{ and } y=\frac{Lr}{2d+r}.$$

The length between the two images seen in the surface and measured with the glass scale is "l" and equals $y\cdot\frac{d}{d+x}$, from which, by substituting the above values of x and y,

$$l=\frac{1}{2}\frac{rL}{d+r},$$

or

$$r=\frac{2dl}{L-2l}.$$

In exactly a similar way is deduced the formula for concave surfaces.

CHAPTER VI

MISCELLANEOUS ELEMENTARY EXPERIMENTS]

(*a*) THE MEASURING MICROSCOPE

THE measuring micrcscope is an instrument of fundamental importance, and therefore its use should be familiar to all students. A very good type of instrument, especially for laboratory work, is made by the Cambridge & Paul Scientific Instrument Co., and is shown in Fig. 81. As

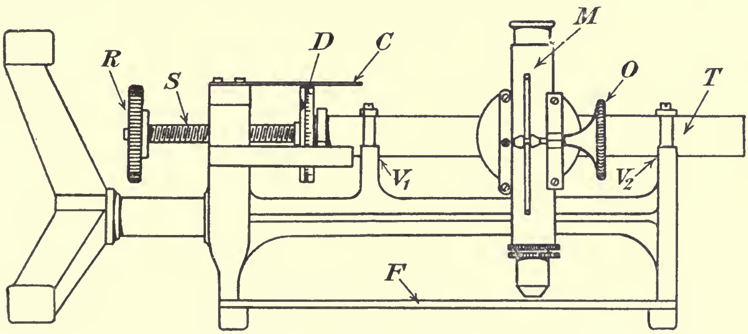


FIG. 81.

will be seen, it consists essentially of a microscope M, which is made to travel by means of an accurate, finely-pitched micrometer screw S. The microscope is attached to the tube T, along which it may be adjusted at will, but can be fixed rigidly when measurements are being taken. The tube T slides in two V's at V_1 and V_2, in which it is held by two opposing springs ; at the end of the tube is situated the micrometer drum D, by means of which intermediate values of the whole divisions on the scale C are read off. Usually C is divided into millimetres and the drum D into one hundred parts, so that with careful estimation readings

may be taken to one-thousandth of a millimetre. F is the stage on which the object to be measured is placed. An advantage of this type of instrument is that it may be used either in a horizontal (as shown in the figure) or a vertical position.

Experiment.—Examine carefully the measuring microscope supplied to you, noting its mechanical construction, the arrangement of the optical parts, and the adjustments, etc., and draw a sketch of the instrument.

Adjust the eyepiece of the microscope to view the "cross-wires" clearly when the eye is "at rest," *i.e.* so that the "accommodation" is not strained. Place the object (a "graticule" or "spectrogram") to be measured on the stage, and carefully focus it by means of the milled head O until the "image" is seen sharply defined at the same time as the cross-wires. Arrange the cross-wires diagonally so that a line of the object may be set accurately on their intersection. In this way measure the distance between consecutive lines of the object by readings obtained from the scale C and drum D. Care should be taken in making a "setting" always to rotate the milled head R in *one* direction for each independent reading, in order to overcome any error due to "backlash" of the micrometer screw.

As an additional experiment, the "pitch" of a screw may be measured in a similar manner. Measurement of (say) three threads will give the interval very nearly; a large number may then be measured without counting, the actual number of threads being found by the first approximate result. The length divided by the number of threads then gives a value for the pitch.

(*b*) APPEARANCES OF "STAR" IMAGE AT THE FOCUS OF A LENS

One of the best ways of testing the performance of a lens or lens system is by viewing the image of a distant star produced by the lens under a high power, such as a microscope or high-power eyepiece.

As actual stars are not always available, a very good artificial star may be made by allowing light from a circular aperture to fall on to a small steel ball about $\frac{1}{2}$ in. in diameter (one from a " Hoffmann " ball-bearing acts extremely well) at right angles to the direction in which the tests are to be made. The extremely small image seen in this spherical surface affords a very satisfactory " point " source. The distance of the lens under test from the artificial star should not be less than 50 feet ; it is advisable also to have a black non-reflecting background immediately behind and in the neighbourhood of the steel ball.

Experiment.—Mount a single lens of about 25 cms. focal length in one of the optical bench lens holders (see Chapter II.), and place it on a one-foot steel rule made up as an optical bench, as described in that chapter. Place a high-power eyepiece (in its fitting) also on the steel rule, direct the latter towards some distant object, and arrange the position of lens and eyepiece until the object is clearly seen. Now direct the optical bench towards the artificial star, carefully centring the system so that the image of the star as seen in the eyepiece appears perfectly central.

Focus the image until it appears at its best focus and make a coloured sketch of what is seen. Then move the eyepiece about 2 cms. *inside* the " best focus " position, observe the appearance, and again draw and colour the rings seen. Do the same when the eyepiece is moved 2 cms. *outside* the " best focus " position.

Explain with a sketch the reason why " *inside* " the best focus a red ring is seen on the edge and blue in the centre, and why " *outside* " the best focus a blue ring is seen on the edge and red in the centre.

Fig. 82 shows what actually happens—light from the star on reaching the lens L is refracted, and exactly as in the case of a prism is split up into its various components, blue it will be remembered being deviated more than red ; so that when the rays are brought to a focus, blue rays will focus at a point nearer the lens than the

red, as shown in the diagram. Consequently when the appearance is viewed inside the focus, as at I, a red ring will be seen on the edge and blue in the centre; and conversely for outside the focus.

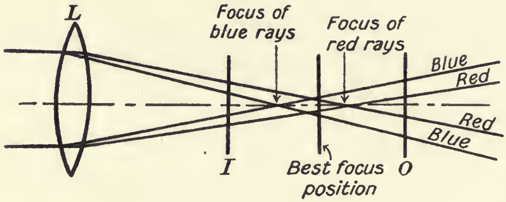


Fig. 82.

This appearance of colours at the focus of a lens or lens system is known as "chromatic aberration."

An "achromatic" lens should now be substituted in place of the "single" lens and the difference in appearance noted. Unless the achromatic lens is an extremely good one the coloured rings will still be detected inside and outside the focus, only on a very much smaller scale, and from these it will be possible to tell whether the lens is "over-" or "under-corrected." In connection with *spherical* aberration, the most noticeable effect seen with an achromatic lens, more especially when a microscope is used to view the star image, is the appearance of a series of concentric dark and light rings; these are due to diffraction. With an "over-corrected" objective the ring system *outside* the focus will be clearer and better defined than that inside the focus. With an under-corrected objective the reverse will be the case. If the lens is satisfactorily corrected the appearance will be the same both inside and outside the focus.

(*c*) DETERMINATION OF THE "FOCAL LENGTH" OF EYEPIECE SYSTEMS

The equivalent focal length of an eyepiece system may be determined very conveniently in the following manner:

The eyepiece to be tested should be held in some suitable mount (a retort stand) on the table at a distance of about 15 to 20 feet from the wall. To the wall should be attached a piece of paper or cardboard on which are painted two bold Indian ink lines about 2 metres apart. If now the eyepiece is directed towards the mid-point of these two lines, images of them will be formed by the eyepiece, as at I_1 and I_2 (Fig. 83), and whose distance

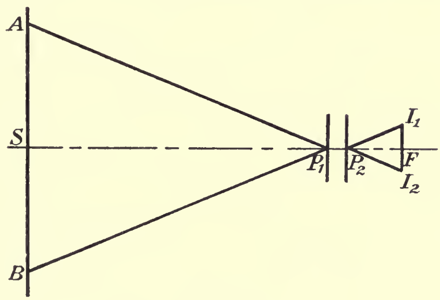


FIG. 83.

apart can be measured either with a "dynameter"* or measuring microscope. In the case of "negative" eyepieces it is better to use the latter with about a 2 in. objective.

In the figure (83) ASB is the cardboard with the two lines at A and B. P_1 and P_2 represent the two principal planes of the eyepiece system, and I_1I_2 the images of B and A. It is at once evident that the triangles ABP_1 and $I_1I_2P_2$ are similar, so that:

$\frac{AB}{SP_1}=\frac{I_1I_2}{FP_2}$. ($FP_2$ is the required focal length.)

$$\therefore\ FP_2=\frac{I_1I_2\times SP_1}{AB}.$$

The distances AB and SP_1 can be measured with string, or preferably a steel tape; if these distances are large, a small error in their measurement will not cause any

* *Dynameter.*—A small piece of apparatus consisting of a Ramsden or positive eyepiece, in the focal plane of which is mounted a finely divided glass scale, usually 1 cm. divided into 100 parts. It proves very useful in many experiments.

appreciable error in the focal length of the eyepiece. The three values on the right of the equation having been obtained, the equivalent focal length of the eyepiece may thus be found.

Second Method.—Another very satisfactory method of determining eyepiece focal lengths is by using a collimator (see Fig. 84) having two lines A and B subtending

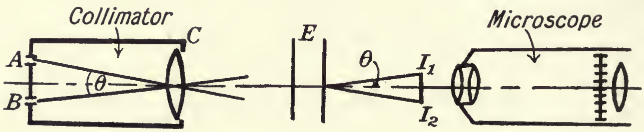


FIG. 84.

a certain angle θ at the object glass C. (This angle is carefully obtained beforehand.) The eyepiece E to be tested is placed in the path of these two parallel beams, and two images are formed at I_1 and I_2 and their separation measured.

Then if I = this separation and "f" the required focal length of the eyepiece,

$$\frac{I}{f} = \theta \text{ (in angular measure, when } \theta \text{ is small, as it is).}$$

$$\therefore f = \frac{I}{\theta}.$$

So that when once I is measured it need only be multiplied by a constant (*i.e.* the reciprocal of "θ" in angular measure), and the focal length of the eyepiece is obtained.

Sometimes a microscope is used to view the image I and which has a scale in its eyepiece. In this case

$$f = \frac{I_3}{\theta \times M},$$

where M is the "first" magnification of the microscope, and I_3 is the separation of the two images measured by the scale in the eyepiece.

A "focometer" of this kind may be very easily made by attaching such a collimator as shown in Fig. 84 to the underside of the stage of any ordinary microscope.

The lens C should be achromatic and about $1\frac{1}{2}$ in. to 2 in. focal length; the two lines A and B should be about 1 mm. apart.

See also Chapter VII., section (*d*).

Searle's Goniometer.—In connection with these experiments a piece of apparatus known as Searle's Goniometer will be found useful. It consists, as will be seen from Fig. 85, of an arm A, on which are mounted a lens L and a single vertical line object O, the latter

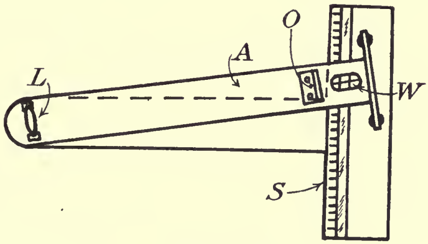


FIG. 85.

being at the focus of this lens. This arm swings about the centre L, and the amount of rotation is read off a scale S by means of a fine wire W. A strip of mirror M is situated at the side and slightly below the scale, in order to ensure a directly vertical observation of the reading being made. This is done by moving the eye until the wire and its image from the mirror appear coincident. So that by the use of this apparatus any angular subtense of the object O may be obtained at will.

As an example, this goniometer may be used in place of the scale on the wall, mentioned in the first method for determining the focal lengths of eyepieces in this chapter.

(*d*) ECCENTRICITY OF A "DIVIDED CIRCLE"

The testing of the eccentricity of a divided circle is always a necessary experiment in order to obtain a knowledge of the error of readings taken from such a circle when in actual use. More especially is this im-

portant when only one vernier is employed on the circle. In the case of more accurate instruments, where micrometer microscopes are used instead of verniers, besides the systematic error brought about by eccentricity, the individual error of each division of the circle must be taken into account. For such a circle a "calibration curve" is made out, so that error for any part of the circle may be read off from the graph.

Fig. 86 will illustrate effects on the readings of the circle due to eccentricity. Let D be the "dividing centre" of the graduations, C the centre of the alidade (*i.e.* the arm on which the verniers are carried), and V_1 and V_2 the zeros of the verniers. Suppose, in this case, that the circle remains stationary and the verniers move round the circle. Evidently, when V_1CV_2 coincides with the diameter through C and D, the readings of the two opposite verniers will differ by exactly 180° (this assumes that the zeros of the two verniers are in one and the same straight line as C), and when at right angles to that diameter the difference will be a maximum. In the figure, V_1CV_2 represents this position, and the "angular eccentricity" will be half the difference in the readings, that is, the angle V_1DA.

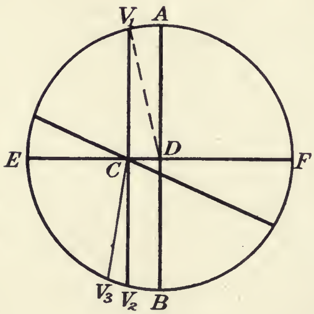


FIG. 86.

Experiment.—The student should be supplied with some instrument fitted with a divided circle with two opposite verniers fitted, such as a spectrometer or theodolite.

Take the readings of the two opposite verniers at twelve or more points round the circle and obtain their differences, care being taken to subtract these values always in the same direction. Then plot the differences on squared paper against the angle; from this the position of zero or minimum departure from the ideal difference

of 180° may be found. In this way a diagram may be drawn showing the relative eccentricity, and a table of values drawn up from the graph, giving the angular error of eccentricity at any point on the circle.

If the difference of the vernier readings is never exactly 180°, the zeros of the verniers and the point of rotation C are not in the same straight line (such as at V_3 instead of V_2). They should be adjusted to be so. This error can be obtained from the graph by the difference between the minimum eccentricity shown and 180° exactly.

Fig. 87 shows typical eccentricity curves. Angular readings of the circle are plotted laterally, and the difference + or − between the two vernier readings are

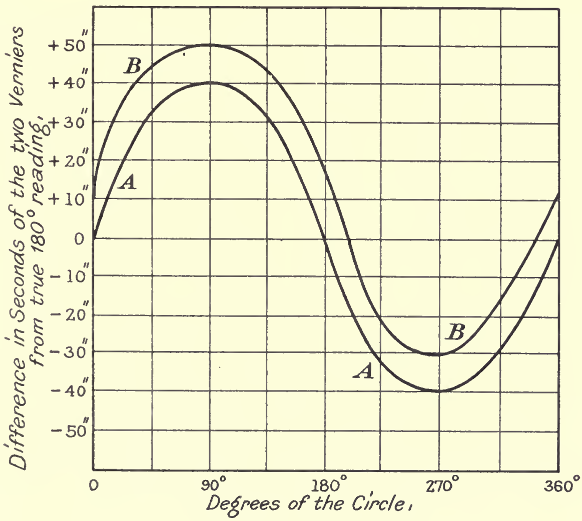


Fig. 87.

plotted vertically above and below the zero position respectively. Curve A indicates that the alidade V_1V_2 (Fig. 86) was coincident with the diameter EF (Fig. 86) at 0° on the circle; and that the greatest eccentricity was at 90° and 270°, and was equal to $\frac{40}{2}$ seconds. Also, that as this error was the same at each of these last two

mentioned positions, the zeros of the verniers must have been set at exactly 180°.

Curve B shows that the greatest angular eccentricity is again $\frac{40}{2}$ seconds, but that as the two exact 180° differences of the verniers occur at 200° and 340° on the circle (*i.e.* not at 180° apart), it indicates that the verniers are not set exactly opposite one another, as illustrated by an alidade V_1V_3 in Fig. 86.

(*e*) PHOTOGRAPHIC TESTS ON A LENS

Apart from the tests for spherical and chromatic aberrations of a lens or lens system, as mentioned in section (*d*) of this chapter, it is sometimes necessary to test the performance of a lens by the actual results given on a photographic plate when a photograph is taken with the lens.

For this purpose the lens should be mounted in some type of camera which has a fairly large " rack adjust-

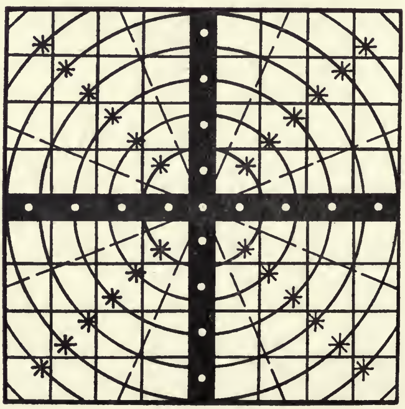


FIG. 88.

ment " for movement of the focussing screen. Two " test-charts " should be made similar to the one shown in Fig. 88, one small and one large. This type of chart is extremely good, as it is designed to bring out every

effect of error that the lens can produce. The size of the two charts depend somewhat on the focal length of the lens under test; the small one can be drawn on a piece of white card with Indian ink, of such dimensions so that when it is placed at the same distance in front of the lens as the image is behind (*i.e.* when $u=v$) the image of the chart will cover the whole of the focussing screen or photographic plate. The second chart will have to be very much larger, as the distance from the chart to the lens in the second case is made about ten times that from lens to image (*i.e.* $u=10v$). It is better if this chart is painted with Indian ink on a flat white wall or board. Card is not advisable, as it is very liable to bend when of large dimensions; and "flatness" is essential.

When these two charts have been prepared, they should be illuminated either with daylight or by a carbon arc and photographs of them taken with the lens. As mentioned before, one photograph should be taken when $u=v$ and one when $u=10v$. The point of making "u" equal to "v" is that defects due to the lens will be more pronounced and are an aid for judging the other result. Of course, the images must be focussed carefully on the ground glass screen of the camera before any photograph is taken; it is best to use a fairly high power eyepiece for this purpose. Exposure should be found by trial—Ilford "ordinary" plates are good for such a test.

When the plates are developed, fixed, washed, and dried they should be examined carefully and the following points looked for:

(1) Central Definition.
(2) Astigmatism.
(3) Distortion.
(4) Coma.
(5) Flatness of Field.

As regards No. 1 (Central Definition), any lack of *sharpness* of the lines (supposing that the plate is at its best

focus position) would indicate that aberrations, either chromatic or spherical, are presented by the lens. It is a good thing to use a yellow screen in front of the lens and so cut out the blue rays, which will to a great extent do away with chromatic aberration.

(2) Astigmatism would be detected by the lack of definition on certain of the " radial " lines at right angles to those on which the definition appeared good.

(3) Distortion, if present, would be most evident at the edge of the plate, where the straight lines of the square would appear curved (a straight-edge should be laid along them). Distortion would be either " barrel-shape " or " pin-cushion."

(4) " Coma " would be indicated by the appearance of the small white circles in the large central cross as being blurred or diffused on one side. Having the effect of a " tail of a comet."

(5) If the definition is equally bad at all four edges of the plate, and if by taking a photograph slightly inside or outside the best focus position for the *centre* of the plate, the definition at the edges improves, roundness of the field would be indicated.

In this way the photographic test on a lens may be carried out; this, combined with the " visual star test " already described for spherical and chromatic aberrations, will give a very good idea as to the performance of the lens.

CHAPTER VII

FOCAL LENGTHS OF "THICK" LENSES AND LENS SYSTEMS

(a) THE "BAR" OPTICAL BENCH

IN connection with experiments dealt with in this chapter it is important that a good type of optical bench be available. The one described in Chapter II. is extremely good for early and more preliminary ex-

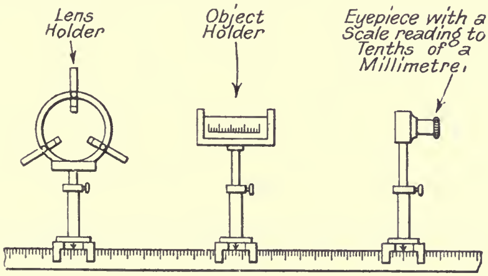


FIG. 89.

periments, but for first-class work it is essential to have a larger and somewhat more serviceable type. Therefore

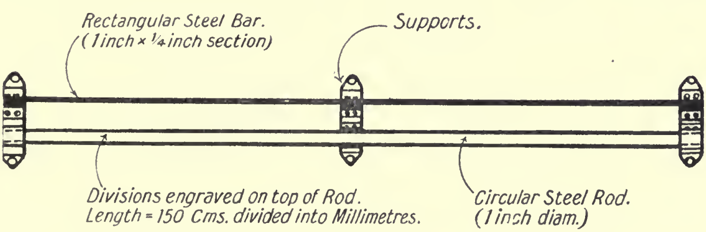


FIG. 90.

it will be well here to describe an extremely good bench which, although not on the market, will be found suitable

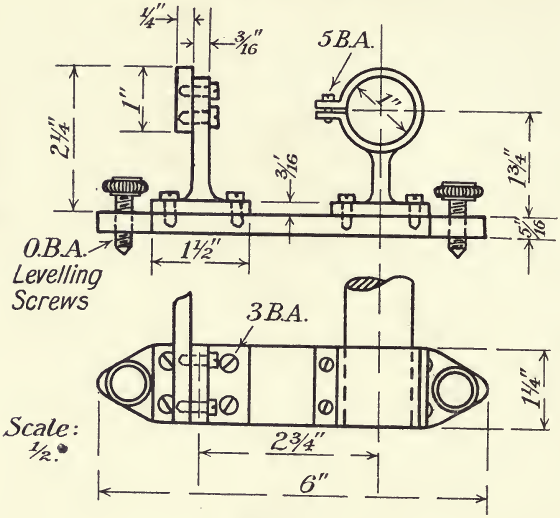


Fig. 91.

for the experiments suggested; therefore scale drawings are given for those who may have the opportunity of making this type of bench for themselves.

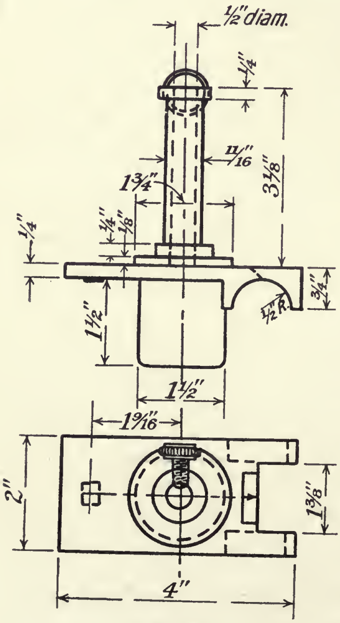


Fig. 92.

Fig. 89 shows the general appearance of the optical bench with the holders and various fittings. Figs. 90 and 91 illustrate rather more clearly the construction of the "bed" of the bench. It consists of a steel rod and vertical bar mounted side by side and parallel to one another, the former being divided in millimetres. Along these two slide the "holders," one of which is shown in Fig. 92; the design of these holders makes them quite rigid and free from any

tendency to turn on a vertical axis when placed on the "bed" of the bench. The cylinder underneath is of lead,

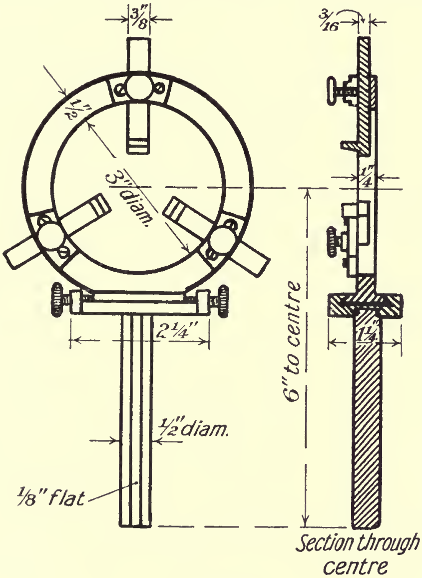


FIG. 93.

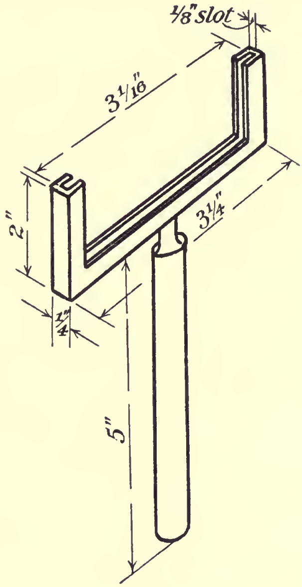


FIG. 94.

in order to make the holder steady. An index line is engraved on a knife-edge on the holder with which readings are taken from the divided rod (constituting part of the "bed" of the bench). The milled head at the top of the hollow "pillar" of the holder is for clamping the stems of the various fittings which fit into these pillars.

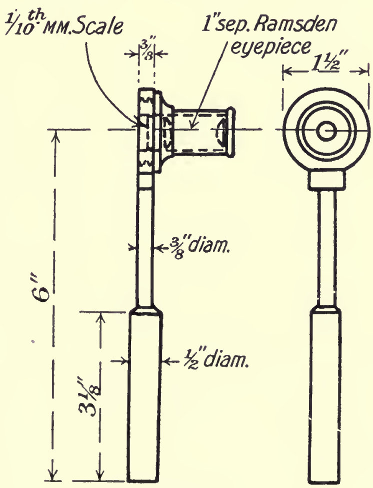


FIG. 95.

Almost any fitting can, in this way, be adapted to the optical bench; the more essential ones, however, are (i) lens carriers, (ii) object

or scale holders, and (iii) a scaled eyepiece ($\frac{1}{10}$th millimetre scale mounted in the focal plane of an eyepiece). Drawings are shown of these fittings in Figs. 93, 94, and 95. Other useful fittings can be made up as desired.

Such a " bench " as this will be found an invaluable piece of apparatus for almost every type of experiment.

(*b*) FOCAL LENGTH OF A "THICK LENS" BY THE MAGNIFICATION METHOD

Revise the theory of the method and prove the following formula :

$$f=\frac{u_1-u_2}{\frac{1}{m_1}-\frac{1}{m_2}}$$

where f = required focal length,

u_1 and u_2 = the readings taken from the optical bench for the two positions of the object scale,

and m_1 and m_2 = the two corresponding magnifications measured with the micrometer eyepiece.

A glance at Fig. 96 explains the formula. The well-known Gauss construction is used, and if this be remem-

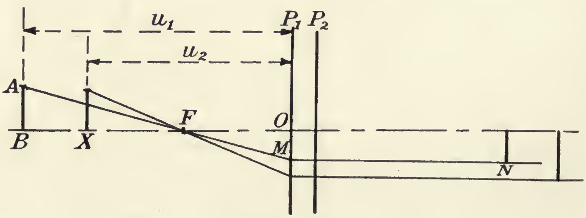


FIG. 96.

bered the formula may be re-derived. P_1 and P_2 represent the two " principal planes " of a thick lens, and AB an object on the left-hand side of the axis. Draw a ray from A passing through the first principal focus F and cutting the first principal plane at M ; then this particular ray will emerge from the lens parallel to the axis, and

the image of A must fall somewhere along MN. Therefore, the size of the image of AB must be MO; so that the triangles ABF and MOF are similar, and thus

$$\left.\begin{aligned} m_1 \text{ (the magnification)} &= \frac{\mathrm{OM}}{\mathrm{AB}} = \frac{\mathrm{FO}}{\mathrm{FB}} = \frac{\mathrm{FO}}{\mathrm{BO}-\mathrm{FO}}. \\ \text{Similarly } m_2 \text{ (the magnification when the object is moved to some other position, as at X)} &= \frac{\mathrm{FO}}{\mathrm{XO}-\mathrm{FO}}. \end{aligned}\right\}$$

These two equations may be written:

$$\left.\begin{aligned} \frac{1}{m_1} &= \frac{\mathrm{BO}-\mathrm{FO}}{\mathrm{FO}} \\ \text{and} \quad \frac{1}{m_2} &= \frac{\mathrm{XO}-\mathrm{FO}}{\mathrm{FO}} \end{aligned}\right\}$$

Subtracting,

$$\frac{1}{m_1} - \frac{1}{m_2} = \frac{\mathrm{BO}-\mathrm{XO}}{\mathrm{FO}} = \frac{u_1 - u_2}{f}.$$

and therefore
$$f = \frac{u_1 - u_2}{\dfrac{1}{m_1} - \dfrac{1}{m_2}}.$$

Experiment.—Place a photographic lens (to be tested) in one of the lens holders on the optical bench. At a distance of about 50 cms. set up a millimetre scale on glass in one of the scale carriers, and then focus the image of this scale on the other side of the lens with a micrometer eyepiece. Measure the size of a number of divisions of the scale as seen through the eyepiece, and thus get the magnification for this position of the object. Make a note of the reading of the "object holder" on the optical bench scale. Move the glass object scale to a fresh position on the bench, focus up the image again and determine the second magnification. From the values obtained and using the formula, the focal length of the lens (f) may be obtained. The experiment should be repeated for a number of positions of the object and the "mean" result obtained.

Negative Lens.—In the case of a "thick" negative lens, or lens system, the same formula holds equally well,

but an auxiliary positive lens has to be used, in the same way as in Chapter II., section (*e*). Form an image I_1 of the scale by means of the positive lens (see Fig. 97)—this serves as the "object" for the negative lens—measure the size of a number of divisions of the scale with the micrometer eyepiece, this value is then the "object."

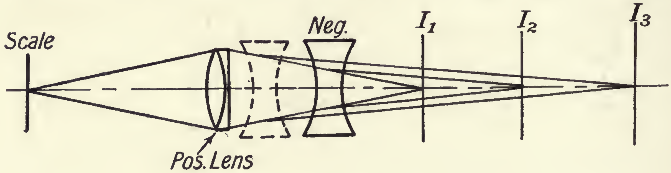


FIG. 97.

Insert the negative lens between the positive lens and this last image, and move the micrometer eyepiece until an image of the scale is again seen (say at I_2, Fig. 97). Measure the size of the same number of divisions of the scale; this value divided by the last will give the first magnification. Then move the negative lens to another position and repeat the procedure. In this way and using the formula the focal length may be determined.

(*c*) "CHESHIRE" FOCAL LENGTH METHOD

A simpler and perhaps more accurate method of determining focal lengths of lens systems has been developed by Professor Cheshire recently. A and B (Fig. 98) are

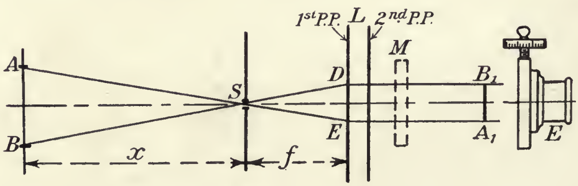


FIG. 98.

two lines of known separation or a millimetre scale on glass. L is the lens to be tested and E the micrometer eyepiece. S is a piece of metal with a narrow (1 mm.) vertical slit cut in it; this piece of apparatus is known as a telecentric

stop and increases the exactness with which A_1B_1 may be focussed. The slit S is set at the first principal focus of the lens under test by placing a mirror M behind the lens and adjusting the latter until a sharp image of the slit is seen reflected back near the "real" slit. When this is the case S will be at the principal focus of L. As the rays AS and BS pass through the first principal focus of the lens, the images A_1 and B_1 must lie on parallel lines DB_1 and EA_1, so that the triangles ABS and EDS are similar, and therefore $\frac{x}{AB}=\frac{f}{A_1B_1}$, from which "$f$" can be found, for A_1B_1 is measured with the micrometer eyepiece, AB is known, and the distance x is obtained by a "measuring rod." A metre or half-metre steel scale set up on the optical bench serves admirably.

This method can be performed very satisfactorily on the "Bar" optical bench described previously.

(*d*) **"FOCO-COLLIMATOR" METHOD** (*Trans. Opt. Soc.*, vol. xxii., No. 1, 1920–21).

This method of determining "focal lengths" is accurate (to ·2 per cent.), extremely simple, but chiefly a quick method. It is this last point which makes the "foco-collimator" very suitable as a "workshop tool."

The principle of the method will be seen from Fig. 99. A and B are two fine diamond lines on glass, situated

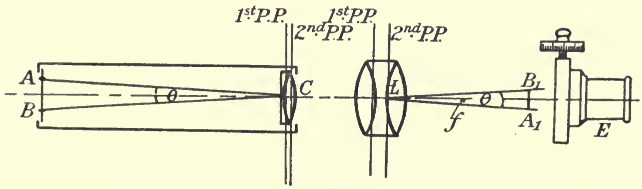


FIG. 99.

accurately in the focal plane of an achromatic lens "C," and subtending a certain definite angle at the first principal plane of this lens. Thus, two parallel beams emerge from the lens inclined at the said angle to another,

so that the lens to be tested L placed in the path of the two beams will form an image of the two lines at A_1 and B_1; and their separation is measured with a micrometer eyepiece E, or "dynameter."

It is quite obvious that the two triangles ABC and A_1B_1L are similar, so that the angle A_1LB_1 = the angle ACB.

Now the angle ACB is previously determined accurately by a method described later, and as A_1B_1 is found by the micrometer eyepiece, it follows, therefore, that the distance "f" (*i.e.* the focal length) may be obtained. For

$$\frac{A_1B_1}{f} = \theta \text{ (in angular measure)}$$

$$\text{or } f = A_1B_1 \times \frac{1}{\theta};$$

but $\frac{1}{\theta}$ is a constant, so that all that is necessary to determine the focal length of a lens is to measure the distance A_1B_1 accurately and multiply it by the previously worked-out "factor." Thus the operation becomes a very quick one and is ideal for the workshop or testing department.

The graticule AB and the lens C, constituting the "foco-collimator," are mounted in metal cells at the ends of a suitable tube and fixed permanently with "set-screws" when finally adjusted. The "multiplying factor" should be engraved on the tube. The lens C should be about 8 in. focal length, and the distance between A and B about 4 mms.

Focussing and Measurement of Angle.—The accurate setting of the two lines A and B in the focal plane of the lens C, and the measurement of the angular subtense of these two lines at the lens, are both of extreme importance. These two settings can be done very completely using the same apparatus in each case. Set up the foco-collimator in a horizontal position and illuminate the graticule from a lamp by means of a microscope cover slip or a piece of mica, as shown in Fig. 100. A mirror M is then placed

as shown, and a microscope (using a 2 in. objective) is arranged to view the graticule. A back reflected "image" of lines of the graticule will thus be produced by the mirror M. It at once becomes evident that when the "image"

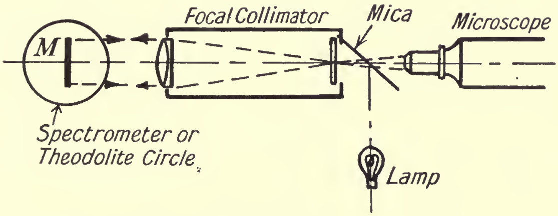


Fig. 100.

of the lines and the "real" lines themselves are in focus simultaneously, as seen on observation with the microscope, the graticule lines must be in the focal plane of the lens C. The distance between the graticule and the lens should be adjusted until correct.

In order to determine the angular subtense of the two lines at C, the apparatus can be used exactly as it is, with the exception that the mirror M should be mounted on the centre of the prism table of a spectrometer or some instrument on which angular rotation of the mirror may be measured. All that is necessary then is, on observing through the microscope, to adjust the mirror until the images of the two lines are exactly coincident with the "real" lines; take a reading of the vernier from the circle on which the mirror rotates, then rotate the prism table (with mirror on it, of course) until the "first" line of the image has become coincident with the "second" "real" line, and read the circle again. This value will give just half the angular subtense.

A Workshop Tool.—A very convenient and useful "tool" for use commercially or in a testing department may be made by a simple adaptation of the principle of the "foco-collimator." It is an instrument for determining the focal lengths of short focus lenses or lens

systems (such as eyepieces) quickly. All that is necessary is to attach a "foco-collimator on a much smaller scale" to the stage of a simple upright microscope. Fig. 101 shows such an instrument in side elevation. C is the small foco-collimator, employing an achromatic lens of about $1\frac{1}{2}$ in. to 2 in. focus and a graticule with the separa-

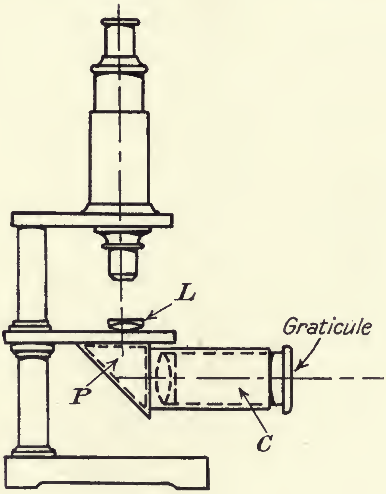


Fig. 101.

tion of the two lines equal to about 1 mm. This is mounted to a metal case which carries a right-angled prism P. The lens to be tested L is rested on the microscope stage, and the images of the two lines formed by this lens are viewed by means of the microscope (which has a tenth-millimetre scale in its eyepiece), with which the separation of the images are measured. Therefore, taking into account the "first" magnification of the microscope (which must be determined beforehand and called here "M"), the separation of the two images will now be :

$$M \times A_1B_1 = \text{(say) } A_2B_2,$$

so that

$$A_1B_1 = \frac{A_2B_2}{M}.$$

Substituting in the previous formula at the beginning of this section, namely,

$$f = A_1B_1 \times \frac{1}{\theta} \text{ (in angular measure);}$$

$$\text{now } f = \frac{A_2B_2}{M} \times \frac{1}{\theta}.$$

$\left(\frac{1}{\theta} \times \frac{1}{M}\right)$ is constant and will be the multiplying factor. So that on measuring the separation of the two lines with the scale in the eyepiece of the microscope it is only necessary to multiply this separation by the "factor" and the focal length of the lens under test is obtained.

(e) "LENS ROTATION" METHOD

This method employs the rotating of the lens system about a vertical axis and can be performed very suitably on the "bar" optical bench. The theory of the method will be seen from Fig. 102. Let N_1 and N_2 be the nodal points of the lens system, which we will suppose has been rotated through an angle θ. Now, a ray AN_1 entering the system and passing through the first "nodal" point will emerge from the lens parallel to its original direction from N_2. If, then, the lens be rotated about any point other than N_2, the ray N_2B will shift from side to side. It is using this fact that the following method is based: A collimator (with either a slit or small circular aperture as object) is set up on one of the "V" supports on the optical bench. The lens to be tested is held in one of the lens holders similar to that shown in Fig. 93, with the exception of a "rack motion" being fitted for movement backwards or forwards of the upper portion of the holder. The image produced by the lens is viewed with a microscope (using 1 in. objective). The lens holder is then rotated through a small angle and back again to the other side, when the image of the slit or aperture (as the case may

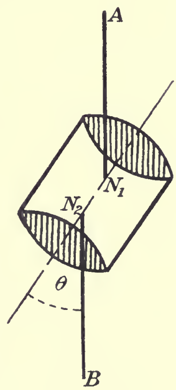


FIG. 102.

be) will be seen to move across or perhaps right out of the field of view of the microscope. The lens should be moved either backwards or forwards by means of the rack motion on the lens holder and again rotated. When the image remains stationary the second nodal point of the lens will be over the centre of rotation of the lens holder. This will be recorded on the optical bench by the index line on the holder. The focal length of the lens will be the distance between this last position and the focal plane of the microscope. This focal plane may be recorded on the bench by resting a set-square on the dividing and moving it backwards or forwards until its edge is sharply focussed when observing through the microscope.

CHAPTER VIII

MISCELLANEOUS ADVANCED EXPERIMENTS

(a) FOCAL LENGTH AND NUMERICAL APERTURE OF MICROSCOPE OBJECTIVE

THE focal length of a microscope objective may be determined in a very simple manner with no apparatus other than a microscope itself, and by the adaptation of an alternative formula used in the "magnification method" of finding the focal length of any ordinary objective (as in the last chapter). The formula is deduced as follows (see Fig. 103):—

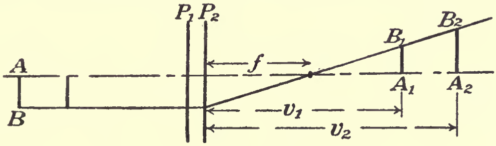


FIG. 103.

A_1B_1 is the size of an "image" of AB produced by a lens at a distance v_1 from the second principal plane of the lens, so that the (first) magnification

$$m_1 = \frac{A_1B_1}{AB} = \frac{v_1 - f}{f}.$$

Similarly, if the image is made to fall at a second position A_2B_2 at a distance v_2 from the second principal plane, the magnification (m_2) in this case will be

$$m_2 = \frac{v_2 - f}{f}.$$

Subtracting, $$m_2 - m_1 = \frac{v_2 - v_1}{f}.$$

$$\therefore f = \frac{v_2 - v_1}{m_2 - m_1}.$$

It is using this formula and the fact that the distance between the two images (*i.e.* the distance $v_2 - v_1$) is required, that the microscope itself can be used for the determination of the focal length, for $(v_2 - v_1)$ can be measured from readings taken on the side of the draw-tube of the microscope.

Experiment.—Screw the micro-objective to be tested in position on the microscope: if a Huygenian eyepiece is fitted, the field lens should be removed as it introduces a slight error in the magnifications. With a Ramsden eyepiece, which should be used if possible, this is not necessary. Whichever type is used, a "tenth-millimetre" glass scale should be fitted in its focal plane for the experiment.

Place a second "tenth-millimetre" scale on the stage of the microscope, draw out the "draw-tube" of the microscope to its full extent and focus this scale. Determine the magnification by estimating the number of divisions in the *eyepiece scale* covered by one or a number of divisions of the "image."

Reduce the tube length by a known amount (say 4 cms. either by taking a reading from a "divided" draw-tube or with a pair of calipers, and measure the second magnification.

Having thus obtained the value for $(v_2 - v_1)$, also m_2 and m_1, "f" may be determined from the formula.

Repeat the experiment for various tube-lengths and take a mean of the values calculated.

(*a*) *Numerical Aperture (N.A.) of a microscope objective.*

Numerical Aperture (usually written "N.A.") is defined as being equal to the product of the refractive index, "n," of the medium immediately outside the objective, and the sine of half the apical angle of the cone of light taken up, *i.e.*

$$\mathrm{NA} = n \sin \text{“}a\text{.”}$$

Numerical Aperture, in connection with the "resolving

power" of a microscope, is even of more importance than the magnification.

Determination of N.A.—This experiment may be performed very conveniently on the "bar optical bench" described in the last chapter. A microscope mounted on a horizontal axis should be placed in one of the holders on the bench. At the extreme end of the bench should be mounted a metre steel rule held in one of the clips on the optical bench. Two pieces of white paper with straight edges should be cut and folded so that they slide conveniently along the edge of the rule. The principle of the method will be seen from Fig. 104. M is the microscope, whose working distance is at A.* CBD is the steel scale with the pieces of paper at C and D. The Huygenian eyepiece of the microscope should be of low power (about 50 m/m sep.), and in place of the ordinary stop a 2 m/m diameter stop should be inserted. The Ramsden circle produced by this eyepiece should be viewed by a positive Ramsden eyepiece placed behind it. The pieces of paper on the steel scale should then be moved outwards from the centre until their edges can only just be seen in the extreme edges of the Ramsden circle. We then have

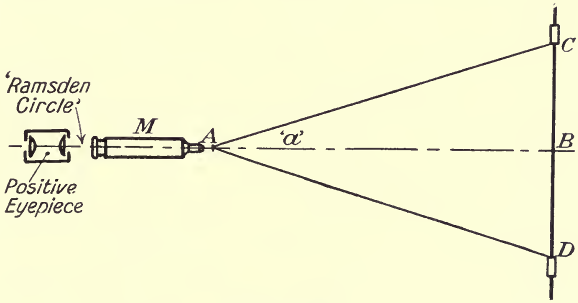


Fig. 104.

* The point A may be fixed relative to the dividing of the optical bench by resting a set-square or tri-square on the latter and moving it backwards or forwards until the vertical edge is seen sharply in focus on observing through the microscope. The reading of the bottom edge of the square may then be taken from the optical bench divisions.

a means of determining the angle "a," for CB or DB (which should be the same) can be obtained from the steel scale and AB from the optical bench. So that

$$\frac{CB}{AB} = \tan \text{ "}a.\text{"}$$

Various distances of AB should be taken for the same objective, the experiment repeated, and a mean value of "a" obtained.

Various quick methods whereby the numerical aperture may be read off "direct" have been devised by Prof. Cheshire, which give very good results. One of these methods consists in placing on the stage of the microscope a piece of card on which is painted the design shown in Fig. 105. This design, when seen in the plane of the Ramsden circle of the microscope, projects as a number

Fig. 105.

of straight lines of equal thicknesses. The distance of the card from the front of the objective is of importance; to obtain this correct distance, a small metal or hard wood block is made of the right length *; this is rested on the card and the top surface of the block focussed with the microscope. The block is then removed, the positive Ramsden eyepiece placed so as to view the Ramsden circle as before, when the number of lines corresponding to the N.A. will be seen just to fill the diameter of the circle.

(*b*) COMPLETE MEASUREMENTS OF THE OPTICAL SYSTEM OF THE MICROSCOPE FOR THE MICROSCOPIST

This section is written for the microscopist who wishes to take measurements on the optical system of his own

* This length should be the distance between the "working distance" of the objective and the position at which the card was calibrated; in this case 25 mms. These cards are obtainable from Messrs Baker, 244 High Holborn.

instrument, and, therefore, naturally does not want to go to the expense of having to obtain much auxiliary apparatus for the purpose.

(*a*) Considering first, then, the *Numerical Aperture* of his objective ; this is best done by using the Cheshire Apertometer * shown in Fig. 105, the method of using being described in the preceding section.

(*b*) *The focal length of the objective* may be determined by the magnification method mentioned in section (*a*) of this chapter, *i.e.* using the draw-tube extension.

(*c*) *The focal length of the eyepiece* can be obtained in a similar way by making up a simple adaptor (see Fig. 105A) to carry the eyepiece, and which can

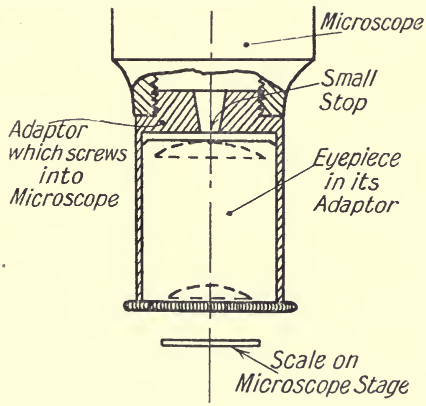


FIG. 105A.

be screwed in position in place of the objective. This is, in effect, using the eyepiece as an objective, which incidentally must be stopped down. Another eyepiece (which has a tenth-millimetre scale in its focal plane) is then used at the eyepiece end of the microscope, and by the magnification method described previously the focal length may be obtained exactly as before.

(*d*) *Magnifications.*—The " first " magnification may be

* See footnote on page 120.

determined by placing a tenth-millimetre scale on the stage of the microscope and comparing the size of the image of a certain number of divisions of this scale, projected by the objective, on a second scale situated in the focal plane of the eyepiece.

The total "magnifying power" may be very conveniently obtained by the method shown in Fig. 105B. A piece of neutral tint glass G (if this is not available, a piece of ruby or cobalt glass will do, or plane glass)

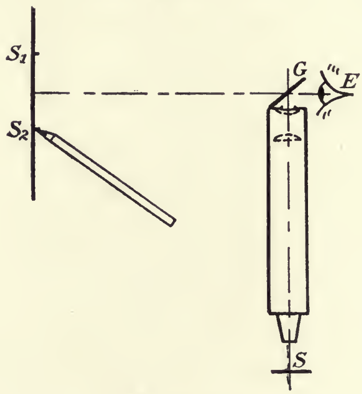


FIG. 105B.

is placed at about 45° to the axis of the microscope. If now the eye is placed at E, the magnified image of the scale S as seen through the microscope can be viewed so that it appears on a piece of card S_1S_2 at about 10 in. away (*i.e.* at the "near point" of the eye). Whilst thus observing, two lines can be drawn with a pencil at the positions where two particular lines of the magnified scale are seen, and then this distance S_1S_2 measured, from which the magnifying power of the microscope may be obtained.

(c) THE AUTO-COLLIMATING TELESCOPE

This instrument is, as its name implies, a combination of a collimator and telescope, and plays an important part in the testing department of the optician. Its applications are many, but it is used chiefly in connection

with the measurement of prism angles and the testing of parallelism of glass plates. It will be well, first of all, to look at the optical system of the instrument; this is shown in Fig. 106. O is the object glass (usually about 12 in. focal length), in the focal plane of which is mounted a graticule G. One of the best types of graticules is that shown in the figure, the horizontal line on the left being

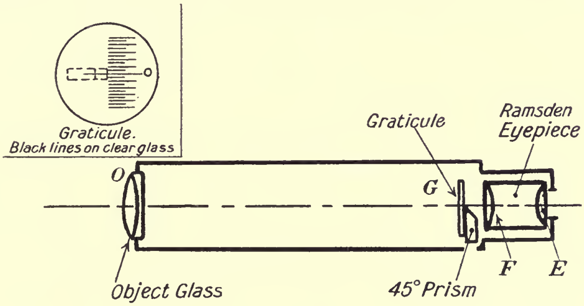


FIG. 106.

covered by a small 45° prism as indicated by the dotted lines, and the spaced lines on the right correspond to a definite angular subtense at the object glass. If a " tenth-millimetre " scale is used, each division may be made to correspond to 1 min. angular subtense by choosing an object glass of suitable focal length, so that by estimation readings may be taken to 6 sec. of arc.

F and E are the field lens and eye lens respectively of a Ramsden type eyepiece. By means of an aperture in the side of the telescope tube light is admitted from a lamp, and thus the previously mentioned horizontal line becomes illuminated. This line serves as the object for the collimator. Rays from this collimator go out " parallel," and if a mirror or plane glass surface is placed in the path of the beam normal to the axis, the rays will return along their original path and come to a focus again in the plane of G, when an image of the horizontal line is viewed by means of the eyepiece. In this way any displacement of the image from the centre line of the scale may be measured in angular amount.

Sometimes an eyepiece with a plane glass reflector in it is used (see Fig. 107), and a graticule of the design

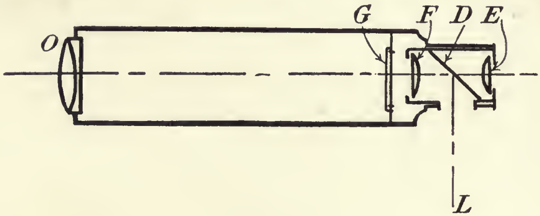


FIG. 107.

shown in Fig. 108 instead of the graticule and 45° prism, but owing to scattered light from the plane glass reflector

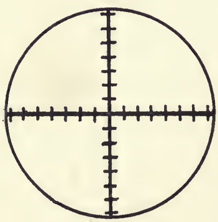


FIG. 108.

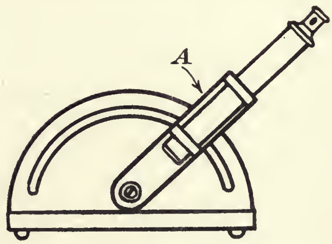


FIG. 109.

it is not nearly so successful as the type of auto-collimating telescope shown in Fig. 106. A very suitable mount and stand for the auto-collimating telescope is shown in Fig. 109; the arm A can be swung into any position within the 180° and can be clamped at will by a "winged" nut at the back of the instrument.

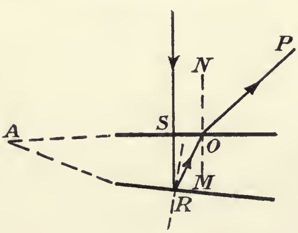


FIG. 110.

Parallelism of a Glass Plate.—If, now, a glass plate is placed in front of the objective, in most cases (unless the two faces are absolutely parallel) two images of the horizontal line will be seen. These are due to reflection from the two surfaces of the plate, the brighter of these two images being the reflection from the first surface. The angular separation of the two images can then be measured on the graticule.

Fig. 110 illustrates the path of the rays in the plate; the angle NOP is the one measured with the auto-collimating telescope, from which, with a knowledge of the refractive index of the glass (it is near enough to take $n=1\cdot5$) the inclination of the two surfaces may be obtained.

For $\angle$SAR (the required angle) $=180° - \angle ASR - \angle ARS$

$$=180° - 90° - \left(90 - \frac{\angle SRO}{2}\right).$$

But
$$\angle SRO = \angle ROM = \frac{1}{n} \times \angle NOP.$$

$$\therefore\ \angle SAR = 180 - 90 - 90 + \frac{\frac{NOP}{n}}{2}$$

$$= \frac{\frac{NOP}{n}}{2}.$$

If n is taken as 1·5

$$\angle SAR \text{ will equal } \frac{\angle NOP}{3}.$$

Testing the Angles of a Right-angled Prism.—The auto-collimating telescope may be used to great advantage for the testing of the angles of right-angled prisms, and becomes an extremely simple and quick method when the observer is once acquainted with his instrument.

It is general to determine the error of the 90° angle first, as this aids the determination of the 45° angles. For this purpose the auto-collimating telescope may be used in two ways: one as shown in Fig. 111A and the other as in Fig. 111B. In the first case, if the angle between the prism face and the "flat" is exactly 90°, only *one* image of the horizontal line would be seen, and that coincident with the "zero" line of the graticule. This, however, is not usual; more often two images will be seen equally displaced each side of the zero. This indicates that there is error in the 90°, and this error (of say a) will be represented by an angular displacement of the two images of "$4a$" on the graticule.

In the second case, when the light travels inside the

prism, the deviation is increased to $n(4a)$, where n is the refractive index of the prism. The student should prove both these for himself.

To test the 45° angles, the auto-collimating telescope

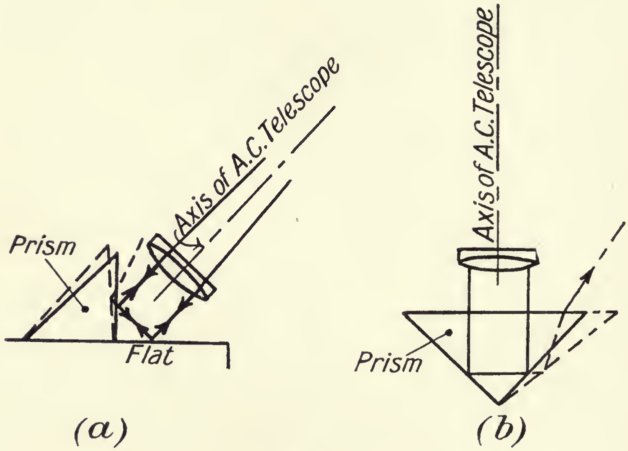


Fig. 111.

should be adjusted until its axis is "normal" to the face AB (see Fig. 112), when the face BC is put carefully in contact with the flat. The prism should then be carefully taken off and the face AC put in contact with the flat. On looking into the telescope, but without altering its position in any way, it will be observed (in all probability, unless the 45° angles are exactly equal) that the horizontal line image has moved a certain number of divisions. This angular movement will be just twice the difference in angle between the two 45° angles. Let this difference be β. Then the $\angle$CAB (supposing that it is the greater of the two 45° angles)

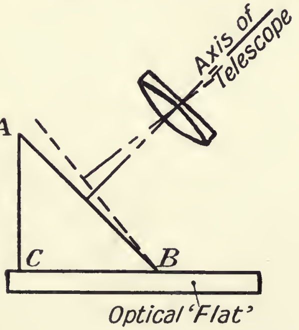


Fig. 112.

$$= \frac{\left(\frac{180^\circ - C}{2} - \beta\right) + \beta}{2};$$

and the $\angle CBA$ $= \frac{\left(\frac{180^\circ - C}{2} - \beta\right)}{2};$

where C is the actual value of the 90° angle.

(*d*) TESTS ON A TELESCOPE

Tests on the performance of a complete telescope are of the greatest importance. They may be divided into two sections:

(i) Geometrical Tests (such as angular field of view, magnification, etc.), and
(ii) Definition Tests.

In dealing with the first section, the focal lengths of the object glass and eyepiece may be determined by methods described in previous chapters. The magnification may be determined very accurately by the following method: Focus the telescope on some very distant object (parallel light). Then support it in a vertical position on the table, with a frosted lamp immediately beneath the object glass. In front of the object glass place one of the millimetre glass scales, and over the eyepiece place a "dynameter" (see page 97), and focus sharply the Ramsden circle, when the divisions of the glass scale in front of the object glass should also be in focus. In this way the size of both scale and image may be measured simultaneously, and the magnification obtained:

$$M = \frac{\text{Size of Scale}}{\text{Size of Image}}.$$

Field of View.—The angular extent of the field of view may be best obtained by observing two distant objects which appear at the extreme edge of the field as seen when looking through the telescope, and afterwards measuring

the angular subtense to the naked eye of these two objects by means of a theodolite or sextant.

If a permanent scale can be set up at some distance (such as might be done in connection with any optical testing department) which has been previously divided according to known angular subtenses, it is possible to read off the angular field of any telescope directly from the scale.

Owing to the (possible) finite distance of the scale, however, it is necessary to place the instrument under test, so that the front or anterior focus of the object glass coincides with the point from which the angular subtense of the scale divisions were previously measured.

Effective Aperture of Object Glass of a Terrestrial Telescope.—The determination of the position and size of the stop

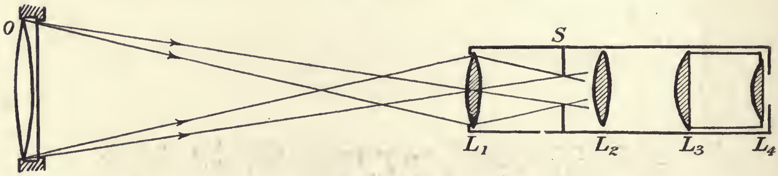


FIG. 112A.

in the "erecting eyepiece" of a terrestrial telescope is of considerable importance, as this is frequently found to be incorrect, with the consequence that the "effective aperture" of the object glass is reduced.

O in Fig. 112A is the object glass, and L_1 and L_2 are the lenses of the "erector." From the paths of the rays proceeding from the object glass shown in the figure, it becomes evident that the stop S must have a definite size and position between the lenses L_1 and L_2 in order to ensure that all the light regularly transmitted by the object glass passes to the image and ultimately to the eye. At the same time any stray light reflected by the sides of the telescope tube are prevented by the stop from passing to and thus confusing the image. Makers frequently take advantage of this point and place this stop in some position such that the definition of their

instrument is increased, but which in effect decreases the aperture of the object glass. This is unfortunate for the customer, who always has to pay for aperture !

To test the Position of the Stop.—Focus the telescope for infinity. Illuminate the aperture of the object glass with a piece of paper and measure the size of the "exit pupil," as mentioned in Chapter VIII., section (*d*). Then take out the complete eyepiece and remove the stop S altogether. Replace the eyepiece and again measure the size of the "exit pupil." If this latter exit pupil (which is the true one) is found to be larger in diameter than the previous one, it is obvious that the stop is cutting off some of the aperture of the object glass. The position of the stop should then be adjusted until the true diameter of the exit pupil is obtained.

(*d*) *Definition Tests* (Test Objects).—For these tests it is essential to have certain definite "test objects." They should preferably be illuminated by daylight, and should be situated at not less than 150 feet from the

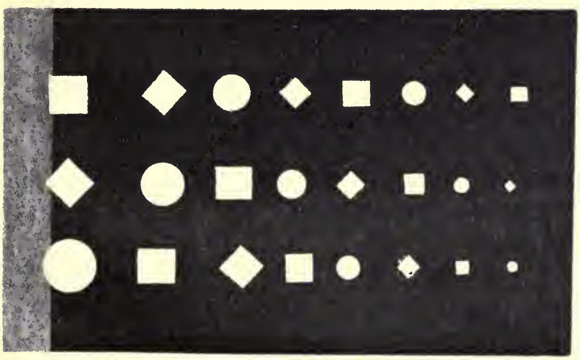


Fig. 113.

position at which observations are to be taken. The most important of these objects is an "artificial star"; this may be made up very easily, as explained in section (*b*), Chapter VI., by employing a small steel ball, on to which light from a circular aperture is allowed to fall, at right angles to the direction in which observations are

to be taken. Such a device gives quite a satisfactory "point source."

The second test object can be made by painting with Indian ink a sketch of a *tree* (without foliage), showing branches and twigs, upon a piece of "opal" glass, and illuminating it from behind. This serves admirably, as the "degree of blackness" of the branches and twigs, as seen through the telescope, serves as an all-important test for the presence of "spherical aberration."

The third object should be one of some such design as shown in Fig. 113. It consists of a metal plate with squares and circles of varying size cut in it. It should be illuminated behind either by artificial daylight or real daylight, in the latter case by means of a mirror at 45°,

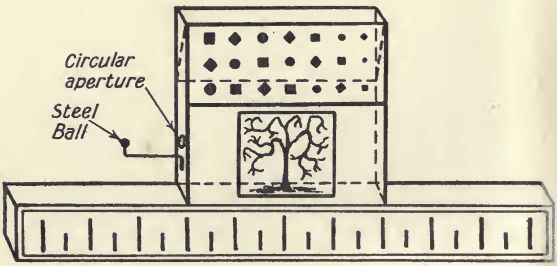


FIG. 114.

and in the former by using a "Chance" artificial daylight screen in front of a ½-Watt electric lamp.

Fig. 114 shows a useful set of test objects which may be mounted together in some form of wooden casing. They should each have a hinged door which can be swung in front at will, in order that any one object may be used without interference from any of the others. Such a set of test objects as illustrated in Fig. 106 is very simple to make, and introduces everything that is essential for telescope testing.

Performance Sheet for a Telescope.—The procedure for testing a telescope will be as follows :

Determine—

(1) Magnification (including size of "Exit Pupil").
(2) Angular field of view.
(3) Set the telescope on the "artificial star" and observe the appearances of the image at the centre of the field :

First. At the best focus.
Second. Inside the best focus.
Third. Outside the best focus.

A properly corrected instrument should show a clearly defined diffraction "ring system" on each side of the best focus. If the rings are "harder" on one side than on the other, "spherical aberration" is indicated. If they are more clearly defined *inside* the focus, "under-correction" of the system is indicated; and if more clearly defined *outside* the focus, "over-correction."

If the rings seen are "elliptical," as shown in Fig. 115, "astigmatism" is present (due probably to some cylindricity of one of the refracting surfaces). "Coma" is identified by the appearance of the ring system shown in Fig. 116.

Colour Correction.—Observations should then be taken on the "colour correction" of the instrument. This is

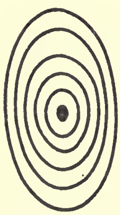


Fig. 115.

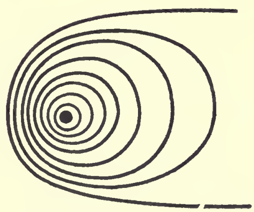


Fig. 116.

best seen by directing the telescope towards the edge of one of the bright large squares shown in Fig. 114. The appearance of this edge inside and outside the focus will be that it is fringed with colour; if it has a *red* fringe *inside* the focus and a blue fringe *outside* the focus, "under-correction" will be indicated. If "red" *outside* the focus and blue *inside*, "over-correction" will be indicated.

A further test for spherical aberration may be given by using the telescope on the black tree test object. Spherical aberration would be indicated by any lack of "*blackness*" of the image seen. For any considerable difference in focus between the marginal and par-axial rays will cause a considerable "scattering" of light, and consequently the image will appear "greyish" instead of black.

The angular extent of "*good*" field should then be determined for each of the above tests, *i.e.* "the star," the bright edge, and the black tree.

Therefore the "Performance Sheet" may be tabulated in the following manner:

Performance Sheet for a Telescope

Form to be filled up for each instrument tested

1. Description of Instrument.
2. Magnifying Power.
3. Angular Field of View.
   (Including size of "Exit Pupil.")
4. Angular extent of "*good*" field.
   First. For "star" object =
   Second. For "bright edge" (*i.e.* colour) =
   Third. For "black tree" object =
   Effective Aperture (stop).
5. "*Artificial Star*" *Tests.*
   (*a*) State whether the system is "over-corrected" or "under-corrected" as regards "spherical aberration."
   (*b*) Is "astigmatism" present?
   (*c*) Is "coma" present?
6. *Colour Correction.*
   State whether the system is "over" or "under" corrected as regards "colour."
7. Centring.
8. General Remarks.

Auto-collimation Test for testing Telescope Objectives.—Another very convenient way of testing the object glass of a telescope is by using the following auto-collimation method. It is in effect a "star" test, using an artificial star by means of light reflected from a small steel ball.

Light from a lamp (Fig. 117) is reflected by means of a steel ball (situated at the focus of the lens), which travels

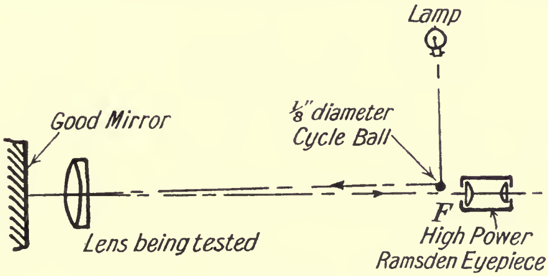


FIG. 117.

towards the lens in the direction indicated. A "good"* mirror is placed as shown and the reflected beam brought to a focus F in the same plane as the steel ball, but slightly to one side of it. The appearances of the "star" image may then be viewed by a high-power eyepiece, and the performance of the lens judged therefrom.

It is of great importance also in this test that the lens be properly "centred" before the star-images may be

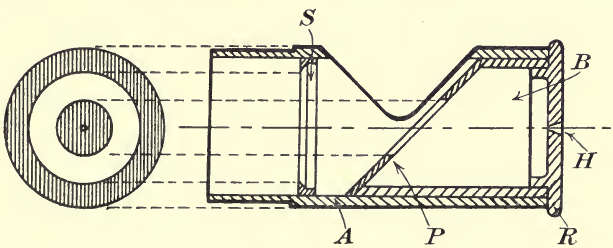


FIG. 118.

fairly judged. For this purpose a piece of apparatus known as a "self-centring eyepiece" should be used.

* It is of the greatest importance that this silvered mirror is a perfectly good "flat."

Such an eyepiece is shown in Fig. 118, and was devised by Prof. Cheshire. It consists of an outer tube A, into which slides the portion B. B consists of a tube, at one end of which is a highly-polished silver or german silver plate P sweated on at 45° with a hole (about $\frac{1}{2}$ in. in diameter) bored centrally in it, whilst at the other end is a knurled ring R which has a pin-hole H drilled in it. S is a stop of the aperture shown.

In use the "eyepiece" is set up approximately on the axis of the object glass to be tested. A lamp is then arranged so as to illuminate the reflector P through the cut-away portion of the tube A, so that an annulus of light will be sent towards the object glass, and to an observer's eye placed at H an effect such as shown on the left of the figure will be seen reflected in the first surface of the object glass. When the object glass is satisfactorily "centred" the annuli thus seen should all appear concentric.

(*e*) TESTS ON PRISMATIC BINOCULARS

The testing of the prism binocular is inevitably an all-important subject, and it is the aim of this "section" to give a complete description of how this should be done.

The tests may be divided up under various headings.

Treating *each half* of the binocular as a telescope :

1. Definition,
2. Magnification,
3. Field of View,

may be determined in exactly the same manner as described in the previous section (*c*).

Other tests are :

4. Parallelism of axes in all positions.
5. Strain in prisms.
6. Inversion produced by the prisms.
7. Stray light.
8. Angular subtense of graticule (if fitted).

Parallelism of Axes.—This is the most important test of all in connection with binoculars. The apparatus

needed for this test is essentially of a somewhat special nature, but as it also serves as a means for adjusting binoculars, it is well worth while having such a device constructed. A diagram of the apparatus is shown in Fig. 119. It consists of two collimators parallel to one another at a distance apart equal to the average separation of the binocular object glasses, an adjustable table on

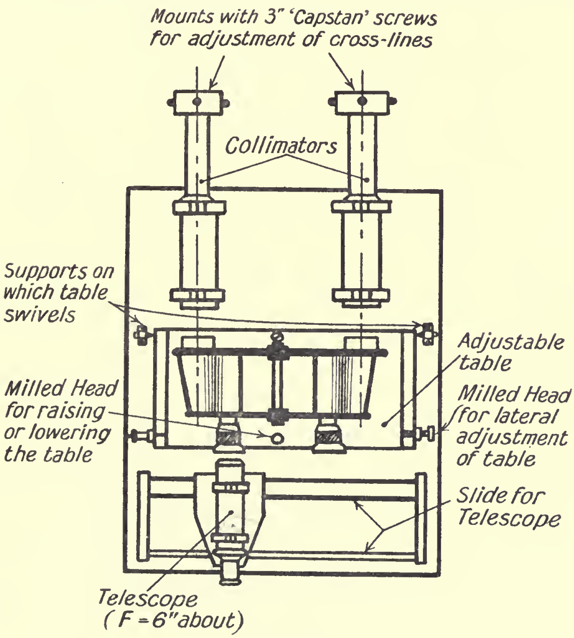


Fig. 119.

which to rest the binoculars, and a small telescope which travels along a geometric slide at right angles to the axes of the collimators (see Fig. 120).

First of all, the axes of the collimators are adjusted parallel to one another by sliding the telescope in front of each collimator object glass in turn, and adjusting the " adjustable " cross-lines of the collimators to coincide with the cross-line in the eyepiece of the telescope. When this has been done the binoculars to be tested are placed on the table and adjusted until the " image " viewed

(with the telescope) through one-half of the binocular is made coincident with the cross-line in the telescope. On sliding the telescope so as to view the "image" through the other half of the binocular, any deficiency in coincidence of the image will at once be seen, and this is a measure of the want of parallelism of the axes of the two

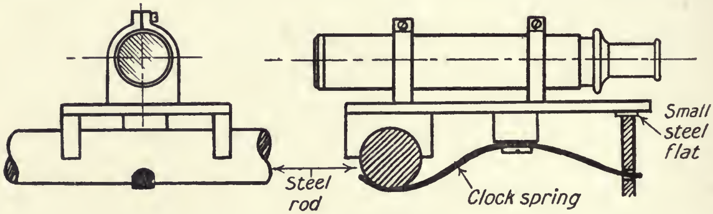


FIG. 120.

halves of the binocular. The actual displacement of the image may be determined in angular amount by the graticule in the focal plane of the eyepiece (a suitable type of graticule is shown in Fig. 108). In this way both the vertical and horizontal angles between the image forming rays from the two halves of the instrument may be determined. Below is given the maximum allowance in angle between the axes of the two halves and the corresponding magnification :

Magnifying Power.	Horizontal Allowance.	Vertical Allowance.
3 ×	30 min.	10 min.
6 ×	12 min.	4 min.
10 ×	6 min. 40 sec.	2 min. 12 sec.
12 ×	5 min. 30 sec.	1 min. 50 sec.

This "binocular testing bench" is also convenient for the ordinary adjusting of binoculars, for the quickly adjustable table allows the binocular to be placed in position and tested with the least amount of trouble possible. Adjustments in binoculars are effected, either by move-

ment of the prisms or by rotation of the object glasses.

Strain in Prisms.—Owing to the method by which the prisms are held in the binocular, excessive pressure is sometimes exerted on them. This is an extremely bad fault, as the double-refracting effect thus produced will appreciably affect the definition of the instrument, and sometimes if the binocular is accidentally given a sharp "jar" a piece of the prism will chip off owing to the strain.

Strain may be quite easily detected by holding the binoculars in a clamp stand and allowing light reflected from a "blacked glass" polarizer (at the polarizing angle, see Fig. 121) to enter them. The "exit pupil" may

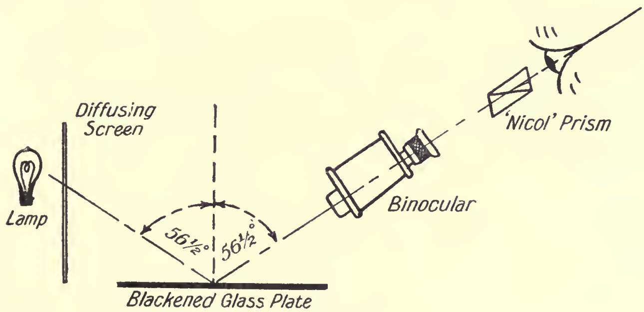


FIG. 121.

then be examined with a Nicol prism, which has been rotated to give "extinction" before the binoculars are inserted in the path of the polarized beam. Any strain in the instrument will be shown up by the appearance of "light patches" among the darkened field. Prisms and lenses of all kinds should be held sufficiently tightly without any undue strain being imposed on them.

Inversion produced by Prisms.—For this test the binoculars are supported horizontally and focussed on a vertical line which is not less than 100 feet away. A theodolite which has been previously made to "transit" over this vertical line satisfactorily (after levelling, etc.) then views

the image of the line through each half of the binocular in turn. In this way the perpendicularity of the image of the line may be estimated, and hence the perfection of the inversion produced by the prisms. Any lack of "inversion" is due to error in the setting of the two prisms at right angle to one another (see Fig. 122).

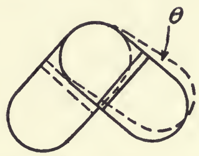


Fig. 122.

Stray Light.—A square frame should be made up, with tissue paper stretched across it and having a black circular disc of paper in the middle, such as is shown in Fig. 123. The frame should be brightly illuminated from behind, and one-half of the binocular directed towards the black disc. The distance of the binocular from the disc should be such that the disc rather more than fills the field of view.

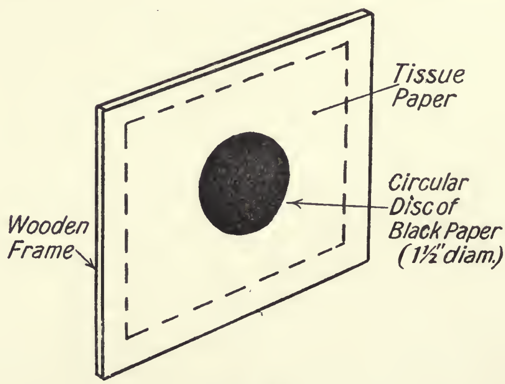


Fig. 123.

On examining the exit pupil, either with an eyepiece or by taking a photograph, any stray light in the instrument will be made manifest. Bright reflections are distinctly detrimental to the action of the binocular, especially in "night glasses."

Test of the Graticule.—A graticule is very often fitted

in the focal plane of one ocular of a binocular, for purposes of "range-taking," and it is necessary that the angular subtense of the graticule divisions (usually 30 mins.) should be tested.

For this purpose the binocular should be supported in a horizontal position, with some source of illumination (preferably diffused) placed in front of the eyepiece. At the object glass end a theodolite should be arranged so

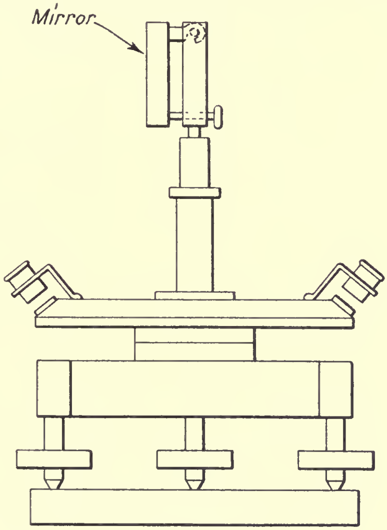


FIG. 124.

as to view the image of the graticule lines. The angular separation may then be measured by setting the cross-wires in the theodolite on the images of the lines in turn and taking readings from the theodolite circle.

Second Method.—Another method may be adopted, which involves the use of a "mirror mounted on a theodolite table"* (a piece of apparatus invaluable to the testing room), and is shown in Fig. 124. Light from a lamp is reflected into the eyepiece of the binocular by means of a plane glass reflector. The theodolite table, with a *good*

* This type of theodolite table without any telescope mounting may be obtained from Messrs E. R. Watts of Camberwell.

mirror mounted on it, is then placed as shown in the figure, and adjusted until an image of the graticule is seen in the same plane as the "real" graticule on observing through the eyepiece. On rotating the theodolite table the image of the central graticule line may be made to travel across the divisions of the real graticule, when readings from the theodolite table may be taken which will give just *half* the value of the actual angular subtense.

CHAPTER IX

REFRACTOMETERS

THE subject of determination of refractive index by the spectrometer was dealt with in a previous chapter, and this instrument in reality is the fundamental instrument for such determinations. There are, however, other instruments designed solely for refractometry which either give "refractive index" direct or by the simple determination of one angle and the use of tables. As these instruments are in considerable use at the present day, this chapter has been devoted to the explanation of the more important types.

(*a*) PULFRICH REFRACTOMETER

The principle of this type of refractometer will be seen from Fig. 125. The substance or liquid whose refractive

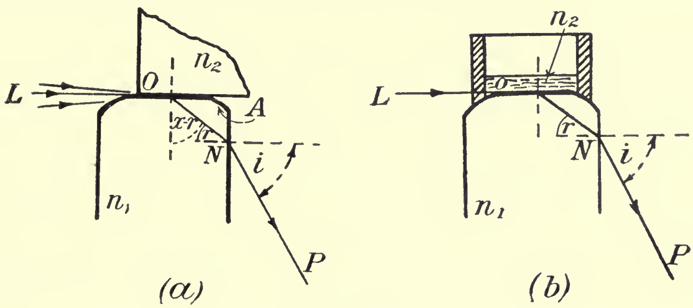


FIG. 125.

index is to be measured is placed on the top of a glass block of known refractive index. In the case of a solid, a thin layer of liquid of high refractive index is placed between the two surfaces. The angle A between the

vertical and horizontal surfaces of the prism is usually very accurately 90°.

If, then, light enters the substance or liquid of unknown refractive index from a position L, that entering above the normal LO will enter the Pulfrich prism and pass out again as indicated along the path NP; a telescope placed at P would then see a band of light with a sharp bounding line on the upper side. The rays which enter normally along LO will graze the two surfaces in contact and will be the limiting rays of the band of light observed at P. Any rays entering below the normal LO will not be able to enter the Pulfrich prism at all.

So that, the sharp line observed in the telescope of the refractometer represents the rays which have just been able to enter the prism; the angle through which these "grazing" rays have been refracted is the complements of $(90° - r)$ . . . . [see Fig. 125], which is the "critical angle" of the Pulfrich block with respect to the substance above it, and depends solely on the refractive indices of the two materials. (An intermediate medium if of greater refractive index than the one above it has no appreciable effect.)

The angle "i" at which the beam emerges into the air depends on the magnitude of the angle "r," and is measured with the refractometer.

Considering the refraction at the two prism faces in turn, we have

$$\left.\begin{aligned} \sin 90 &= \frac{n_1}{n_2} \sin (90° - r) \\ \text{and } \sin \text{``}i\text{''} &= n_1 \sin \text{``}r\text{''} \end{aligned}\right\}$$

where n_1 and n_2 are the refractive indices of the Pulfrich block and substance or liquid to be tested respectively.

Combining these equations, the unknown refractive n_2 is calculated from the expression

$$n_2 = \sqrt{n_1^2 - \sin^2 i}.$$

The instrument is usually supplied, however, with a

table which is prepared for all values of the angle "i" from the above formula, so that refractive index may be determined directly from the table.

Fig. 126 gives a general illustration of the instrument. Light (usually from a hydrogen tube) is sent into the substance or liquid being tested, through the condenser "C," which renders the light convergent. The beam on

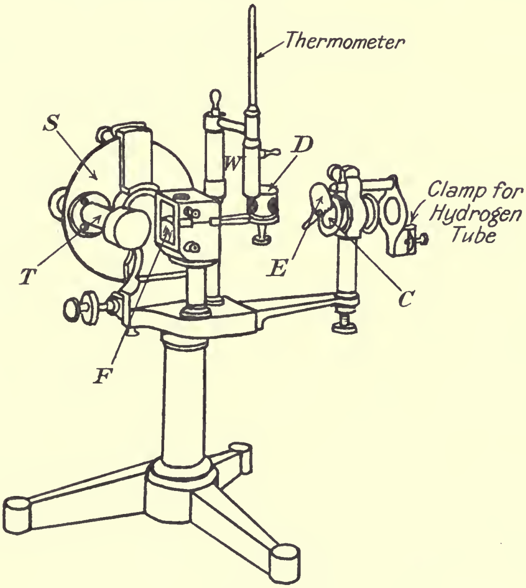


Fig. 126.

emerging from the Pulfrich prism face F is received by the telescope T, which is attached to the rotating circle S. With this circle, by means of a vernier and tangent screw the angle "i" is measured. The telescope is auto-collimating, in order that the normal to the face F of the Pulfrich block may be obtained by back reflection. Fittings W for a water circulation are provided, so that substances may be investigated at raised temperatures; also it is particularly useful in the case of substances, such as fats and waxes, which only become liquid and transparent at these temperatures. D is a right-angled

prism which can be swung in and out of the path of light from the condenser so that sodium light may be used when desired without having to remove the hydrogen tube. E is simply a device for limiting the aperture of the incident beam.

The Instrument in Use.—When testing a solid it is essential that the specimen has two surfaces nearly at right angles, and the one which is placed in contact with the Pulfrich prism should be well polished and reasonably flat, while the other need only be sufficiently polished to allow light to enter. It is important, however, that the edge at which the two surfaces join should be very sharp. For measurements on liquids a small glass cell is cemented on the top of the Pulfrich block,* into which a small amount (a layer of about 3 mms. deep) of the liquid can be held (see Fig. 125 (*b*)).

First of all, the reading of the circle when the telescope is "normal" to the face of the "block" should be obtained (*i.e.* the zero setting checked). For this purpose a lamp should be arranged to illuminate the small prism near the eyepiece of the telescope, and the circle rotated until an image of two small lines will be seen in the field of view. When they are in such a position that the *one* line of the real graticule is midway between them, this should give the zero setting of the instrument. The type of graticule generally used is shown in Fig. 127.

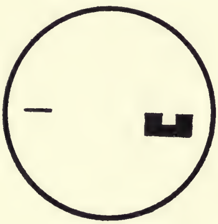


Fig. 127.

This being done, the specimen to be measured should be placed on top of the "block." This has to be done with great care. The two surfaces to be put in contact should first be *thoroughly* cleaned. A small quantity of liquid † (of higher refractive index than either the specimen or the Pulfrich prism) is then placed between the two

* Two Pulfrich prisms are supplied with the instrument, a "light" and "dense," the former for use with liquids and the latter for solids.

† A suitable liquid to use, and of high refractive index, is "Monobrom-naphthaline."

surfaces, and the specimen pressed firmly on to the prism. The surface of contact should then be examined by reflecting from it monochromatic light (a sodium flame), when alternate light and dark interference bands will be seen. The bands should be made as broad as possible by pressing on the specimen, as the surfaces are then most nearly parallel. The number of bands seen should not be more than six, and with "well-worked" specimens having flat surfaces it will be found possible to bring the surfaces so parallel that one band fills the whole surface of contact.

The instrument is now ready for taking readings. Sending "sodium light" first, therefore, through the specimen, the telescope should be moved round until the graticule is brought on to the sharp bounding edge between the sodium coloured and dark part of the field. The difference in readings taken at this position and that of the "normal" or "zero" reading will give the value "i" (Fig. 125).

By a simple reference to the tables supplied with the instrument, the refractive index (for D light) of the specimen can be obtained corresponding to the value of "i."

Similarly, by using the hydrogen tube* with the instrument the values of "i" may be obtained for the C, F, and G_1 lines, and with the use of the tables the refractive index for each line. Also mean and partial dispersions may be obtained.

Tables are also supplied for temperature variation when such are needed.

(*b*) THE ABBE REFRACTOMETER

The principle of this instrument again depends on the use of a standard prism and the border line between the

* The most suitable hydrogen tube to use is the type mentioned in Chapter IV., as the large side bulb allows a much heavier current to be put through the tube without the great rise in pressure, and thus increases the intensity of the G_1 line.

K

light and dark parts of the field, due to "grazing incidence" illumination.

Its general arrangement will be seen from Fig. 128. It consists of two 30° prisms A and B, mounted in a metal casing which can be rotated on a horizontal axis immediately beneath a telescope T. To this metal case

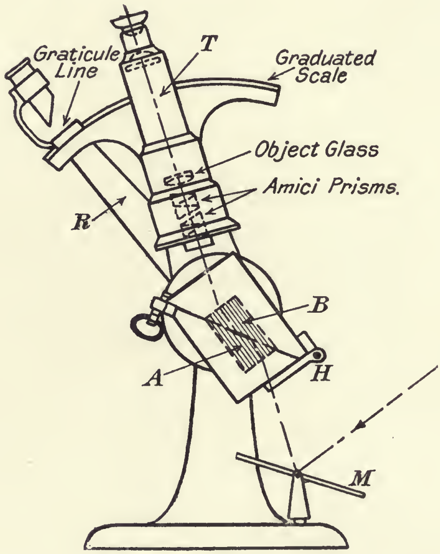


FIG. 128.

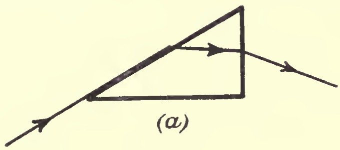
is attached an arm R, at the end of which is a graticule line; this moves over a scale graduated directly in terms of "refractive index."

The general principle of the use of the standard prism is the same as in the case of the Pulfrich refractometer, but it should be noted its angle is no longer 90°. B in Fig. 117 is the standard prism and is usually of "dense flint"; the auxiliary prism A is solely for the purpose of leading light at "grazing emergence" into the liquid film, when of course it will fall on the main prism face at "grazing incidence."

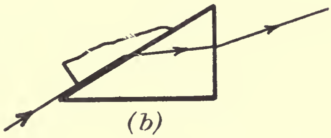
Fig. 129 (*a*) shows the path of "grazing incidence" light on the face of the standard prism; Fig. 129 (*b*) shows the

use of a prism when testing a solid or when using a test prism; and Fig. 129 (*c*) shows the prisms as generally used when the liquid being measured is spread out as a thin film between the flat glass surfaces.

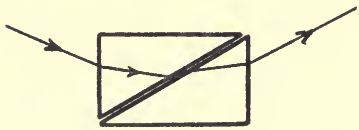
Light is admitted into the prism system by means of a mirror M (Fig. 128). This may be either light from the



(*a*)



(*b*)



(*c*)

Fig. 129.

sky or from a lamp; monochromatic light is not necessary, as the colour of the "bounding line," as seen in the telescope, is annulled by the use of two "direct-vision" prisms (known as Amici prisms), situated in front of the object glass, and which can be rotated in opposite directions by means of a rack and pinion. Only when the "bounding line" is properly achromatized can readings be taken.

These Amici prisms (see Fig. 130) are so constructed that they have no deviation for "D" light, but will produce deviation for all other colours; so that two such prisms relatively inverted will be achromatic, but similarly placed will produce approximately double the dispersion due to one alone. For details of construction a text-book on Geometrical Optics should be consulted.

The exact calculation of the dispersion due to two such prisms when placed at any relative angle is a very awkward one, and it is probably better to calibrate the instrument experimentally. It should be remembered, however, that the dispersion value furnished by this test is only approximate and is only meant for rough identification purposes.

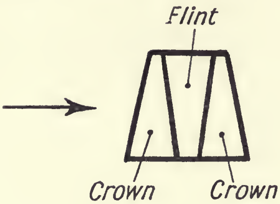


FIG. 130.

The Instrument in Use.—The two surfaces of the prisms between which the test liquid is to be put should be thoroughly cleaned. The instrument is then swung into such a position so that the hypotenuse face of the standard prism is horizontal; a few drops of the liquid are then put on, and the other prism in the other half of the metal case swung over and clamped. The telescope of the instrument should then be brought into its most convenient position, and on looking in through the eyepiece the mirror M (Fig. 128) should be adjusted until good illumination is present in the field. The arm R should then be moved round until the "bounding line" between the light and dark part of the field comes into view. This must then be made quite free from any colour fringes by rotating the Amici prisms by means of the milled head provided for that purpose. The cross-wires in the eyepiece should be sharply focussed and the bounding line set accurately on to the intersection of the former. The reading given by the graticule index line on the graduated scale will give the "refractive index" of the liquid. The scale is graduated from 1·3 to 1·7 and is divided to the third decimal place of refractive index, the fourth place being obtained by estimation. As with the Pulfrich refractometer, temperature precautions are of the greatest importance with liquids, and therefore the prisms are surrounded with a water-jacket to secure constancy in this respect. If available, a "thermostat" should be used to ensure uniform circulation of the water.

(*c*) REFRACTOMETER FOR GASES

The determination of the refractive indices of gases is obviously a more delicate operation than that of liquids and solids. To obtain the required sensitiveness of the instrument a method employing "interference" of two beams of light is used. The original principle was from Lord Rayleigh, but the instrument described here is a modification of this principle. A diagrammatic sketch of the optical system is shown in Fig. 131. Light from a small electric lamp L illuminates a slit "S," which is in

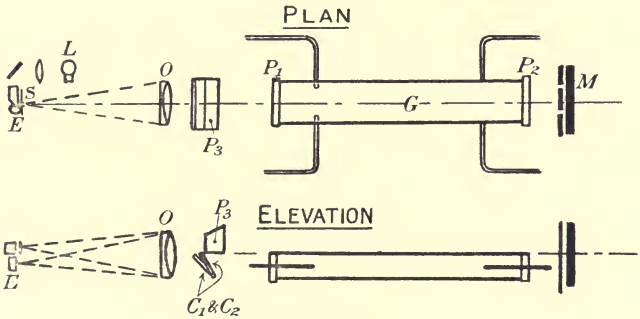


FIG. 131.

the focal plane of an achromatic lens O (about 6 in. focal length). The parallel beam emerging from this objective travels on until it reaches a mirror M, which has two parallel slits of known separation in front of it. The light then returning along the same path will come to a focus again in the plane of the slit S, where a bright image of the slit will be seen with a number of diffraction bands on either side of it. Two gas cells G (of known length) side by side are situated in one half of the parallel beam, as illustrated between the object-glass O and the mirror M, so that half the beam travels through the cells and the other half over the top of them. C_1 and C_2 are two "compensators," one of which is adjustable by means of a slow motion with micrometer screw. P_3 is a block of glass situated in the top part of the beam and equal in thick-

ness to the combined thicknesses of the plates P_1 and P_2 and one thickness of a compensator C_1; this is done in order that both halves of the beam shall travel in equal amounts of glass. The appearance, therefore, as seen with the eyepiece E,* will be that of two sets of diffraction bands, one set from the lower half of the beam, the light of which will have passed through the gas cells and back again; whilst the other set will be formed by light passing over the top of the cells and back again.

When the instrument is adjusted correctly the central bright band of light of the lower set of bands should be coincident with the central bright band of the upper set. If, now, the gas to be tested is allowed to flow through one of the gas cells, and air is still kept in the other, any

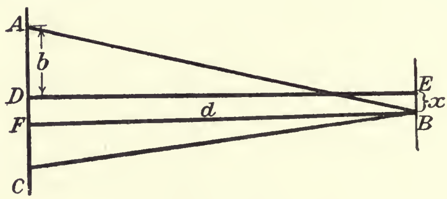


FIG. 132.

change in refractive index will be represented by a retardation of the beam through the gas cell. This retardation in one of the halves of the lower beam will result in a lateral displacement of the lower set of diffraction bands. The upper set of bands, of course, will not move at all, and thus they serve as a reference mark. By bringing the central bright band of the lower set of bands back to its original position by tilting the compensator C_1 with the micrometer slow motion, the retardation produced by the gas cell may be calculated for (see Fig. 132).

Let AC be the second principal plane of the object glass and also the distance apart of the two slits, DE the focal length of the objective; and let B be the position of the

* The eyepiece is made up from a piece of cylindrical glass rod about 3 mms. in diameter; this gives a very high magnification, which is necessary in order to be able to see the diffraction bands at all.

central bright band of the lower set of bands, after being displaced a distance EB.

Then for a bright band to be formed at B, the difference in path between AB and BC must be an even number of half wave-lengths. This difference in path is obtained from the following :

Call $AD = b$, $FB = d$, and $EB = x$,

then $AB^2 = d^2 + (b + x)^2$,

and $BC^2 = d^2 + (b - x)^2$.

Subtracting $AB^2 - BC^2 = 4bx$,

or $(AB + BC)(AB - BC) = 4bx$;

but $(AB + BC) = 2d$ (sufficiently near).

So that $AB - BC = \dfrac{2bx}{d}$.

From this the actual retardation of the beam passing through the gas cell may be obtained ; with this and a knowledge of the length of the gas cells,

$$n = \frac{\dfrac{2l}{\lambda} + \dfrac{(AB - BC)}{\lambda}}{\dfrac{2l}{\lambda}}.$$

where n = the refractive index,

l = the length of the gas cells,

λ = the wave-length of light.

AB and BC = the distances referred to in Fig. 132.

Pressure and drying precautions of both air and gas should, of course, be taken.

CHAPTER X

APPLICATIONS OF POLARIZED LIGHT

THE theory of the subject of polarization should be revised from other text-books, as this chapter deals with useful applications of polarized light.

(*a*) DETECTION OF STRAIN

One of the most convenient ways of producing a beam of "polarized" light is by reflection. If skylight is reflected from a blackened glass plate so that the reflected

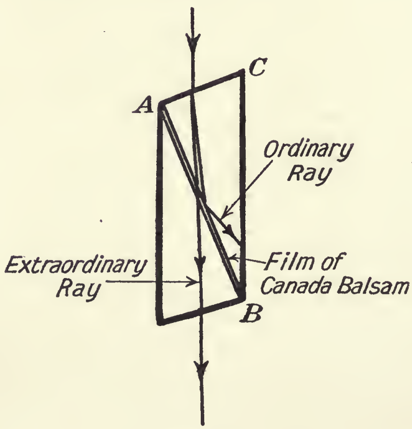


FIG. 133.

beam leaves the plate at the correct angle (viz., $56\frac{1}{2}°$ with the normal), the light thus reflected will be plane polarized.

Another very usual method is by employing a "Nicol" prism. Such a prism is shown in Fig. 133; it consists of a rhomb of Iceland spar cut and cemented together

along the face AB with Canada balsam. A ray entering the face AC will be split up into its two components, the "ordinary" and "extraordinary" rays, the former of which has a greater refrangibility. Canada balsam having a refractive index between that of the ordinary and extraordinary rays, and the length of the prism ABC being suitable, the "ordinary" ray is "totally reflected" at the face AB, whilst the extraordinary ray passes almost straight on and leaves the prism with its original direction. So that there will no longer be two beams coming out of the spar with vibrations at right angles to one another, but one beam with vibrations only in one direction or plane. Thus the Nicol prism is a very suitable means of obtaining plane polarized light.

The double refracting effect produced by Iceland spar is also present when ordinary glass is under any stress, due either to applied pressure or to bad annealing of the

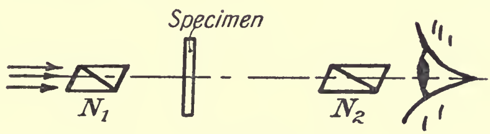


Fig. 134.

glass setting up internal stresses. Such strain becomes very evident when a suspected specimen is examined by polarized light. It is this fact that makes the use of polarized light of such importance. Suppose two Nicols N_1 and N_2 (Fig. 134) to be "crossed" so that all light is extinguished, and let the specimen to be tested be placed in between the two. If any strain is present it will be represented by the appearance of patches of light in the previously dark field, and colour effects will be seen when great pressure is present.

A convenient piece of apparatus for detecting strain may be arranged as shown in Fig. 135. Light from the sky or from a *diffused* artificial source strikes a blackened * glass reflector B. The reflected beam is then viewed with

* Frigelene varnish.

a Nicol in front of the eye. On rotating the Nicol and by moving the head up or down a position will be found when the reflected beam is almost entirely extinguished. The Nicol should then be rigidly held in this position with a clamp of some kind. The object to be tested is then placed between the reflector and the Nicol

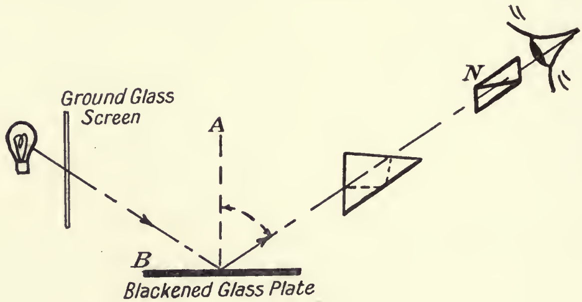


FIG. 135.

as shown, when any strain in the specimen will at once be detected. If a piece of glass is held in a small vice, placed in the beam as before and the vice gradually tightened, the effects due to increased pressure will at once become obvious.

Almost any optical work, both mounted and unmounted, may be examined for strain in this way. More especially is this test essential to ascertain whether object glasses are held too tightly by their counter-cells, the *over*-clamping of prisms, and numerous other cases.

(*b*) MICROSCOPE POLARIZER

A simple application of the blackened glass reflector is to form a polarizer for the microscope, for use in connection with petrological work. For observation of rock sections, etc., polarized light is greatly advantageous in the microscope. In place of the somewhat expensive Nicol prism usually used as the polarizer, a 3 in. × 1 in. cover slip may be blackened with varnish and stuck with soft wax to the tilting mirror beneath the microscope

stage. On observing through the analyser and rotating same the tilt of the slip polarizer can be adjusted until

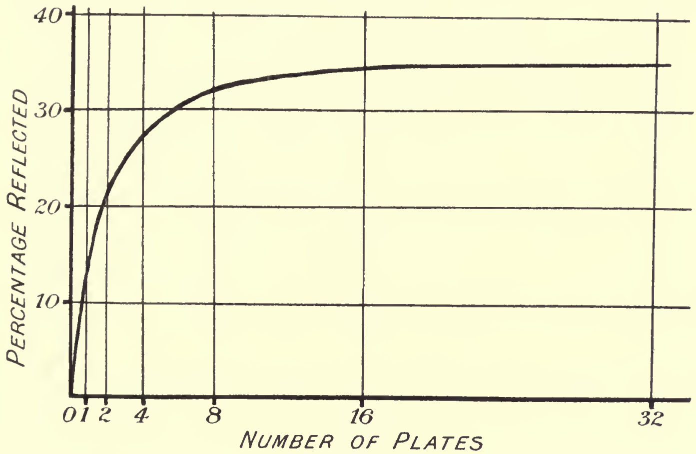


FIG. 136.

the best position of "extinction" is obtained. This makes quite an efficient polarizer. The polarization of

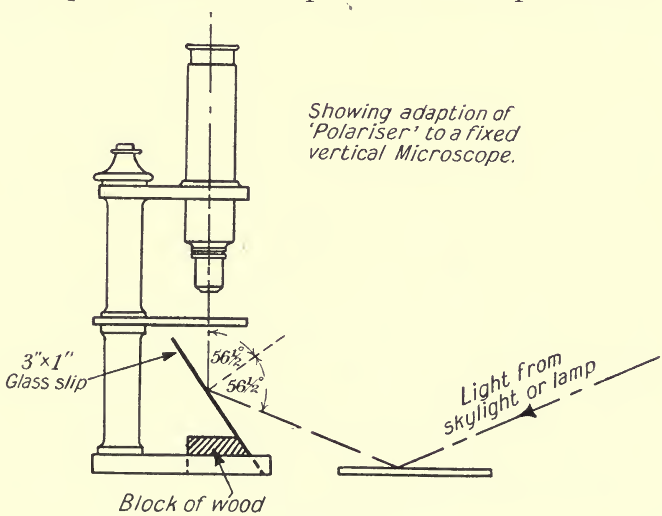


FIG. 137.

the beam, however, may be increased if necessary by superposing a second 3 in. × 1 in. slip on the face of

the first. Fig. 136 shows a graph by Stokes giving the relationship between the percentage of polarized light reflected from a number of plates from 1 to 32, the light being incident at the polarizing angle (*i.e.* $56\frac{1}{2}°$). From this curve it will be seen that practically the maximum amount of polarized light which can be obtained by reflection is from eight plates. In practice, however, one, or at most two, will be sufficient for most work. Fig. 137 shows a "slip-polarizer" used with a simple vertical microscope, where it is necessary to use an auxiliary mirror lying flat on the table in order to get light into the instrument.

(*c*) SACCHARIMETERS

One of the most important applications of polarized light at the present day is saccharimetry.

Certain transparent substances possess the property that when plane polarized light is passed through them it emerges plane polarized, but in a different plane to that of polarization at incidence. These substances are said to rotate the plane of polarization; such a substance is quartz. The effect is also produced by solutions of certain substances; for instance, a solution of sugar in water rotates the plane of polarization. The rotation which a substance produces is the key to the determination of the degree of concentration of that substance in solution. The instrument for measuring this rotation is known as a saccharimeter or polarimeter; they are used to a very great extent commercially in testing "sugar" solutions.

It has been determined that the rotation produced is proportional to the "mass" of substance in a given volume of the solution.

Now, suppose a mass, "w," of a substance to be contained in each cubic centimetre of an inactive solvent (*i.e.* one that does *not* rotate the plane of polarization), and let plane polarized light of a definite wave-length

(say sodium) traverse a length "l" of the solution. Then the rotation R is proportional to "lw."

So that $$R = Klw \qquad \text{(i)}$$

where K is known as the "specific rotation" of the substance. K is dependent to some extent on the wave-length of light, also temperature and concentration. To determine K by experiment, suppose "x" grammes of the substance to be dissolved in "y" grammes of the solvent, and let the density of the solution be "d." Then, the volume of the solution is

$$\frac{y+x}{d} \text{ ccs.}$$

Therefore, the mass of the substance contained in a cubic centimetre is:

$$x \div \frac{y+x}{d}$$

$$= \frac{xd}{y+x}.$$

Thus (from (i)) $$R = \frac{Klxd}{y+x},$$

and $$\therefore K = \frac{(y+x)R}{lxd} \qquad \text{(ii)}$$

So that, from formula (i) it is clear that if for any substance the value of K is known, and the rotation R, produced by a known length "l" of the solution, can be determined, the concentration "w" of the solution may be obtained.

The optical system of a very usual type of saccharimeter is shown in Fig. 138. The source of light S is placed at the focus of the lens L_1, so that a parallel beam of light enters the polarizing Nicol N_1. Two auxiliary Nicols A and B (known as Lippich prisms) are situated immediately behind N_1. Thus the field as seen with the telescope consists of three parts: the central part N_1 corresponds to light which has passed through the analyser

only, while the two outer parts A and B correspond to light which has passed in addition through the two auxiliary Nicols.. It is found that dividing the field into three parts in this way facilitates the accuracy with which the Nicol N_2 can be set. N_2 is the second Nicol which together with its mount rotates with the divided circle C, from which readings of rotation are taken. The solution to be investigated is placed between the polarizer and analyser; it is enclosed in a tube of known length,* at the ends of which are plates of optically worked glass.

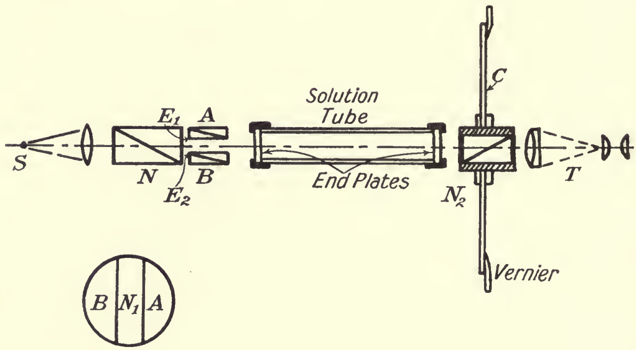


Fig. 138.

These plates are held against the ends of the tube with metal caps, so that the plates may be removed for cleaning and filling the tube. T is a low-power telescope which focusses on the sharp edges E_1 and E_2 of the auxiliary Nicols, thus giving a sharp dividing line to the three parts of the field as seen through the telescope.

"*Soliel*" *Type.*—In some forms of saccharimeter the angle through which the plane of polarization is rotated is measured by interposing a certain thickness of some substance which rotates the plane in the opposite direction, and thus neutralizes the rotation produced by the solution under investigation, instead of measuring the rotation with a divided circle.

* They are usually 10, 20 or 30 cms. in length.

Soliel devised a means involving the use of two quartz wedges ABC and DEF (Fig. 139), which by means of a rack and pinion are caused to move in opposite directions, thus enabling varying thicknesses of quartz to be obtained. The wedges have equal angles and are cut with the optical axis of the quartz perpendicular to the faces BC and DF.

When the wedges are immediately behind one another, a scale mounted above should read zero. The solution to be tested is then placed in the instrument, and the wedges moved so as either to increase or decrease the thickness of quartz until the two halves of the field (*i.e.*

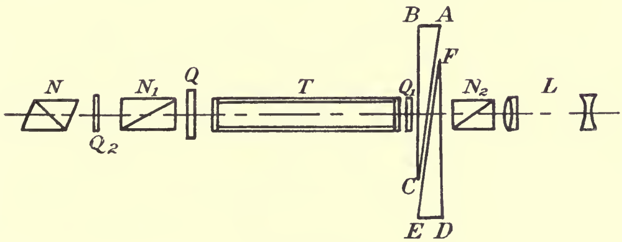


Fig. 139.

when using a bi-quartz) again appear equally dark. The reading from the scale can then again be taken. The scale is calibrated beforehand with solutions of known rotation, so that any reading on the scale may at once be converted into angular rotation from the graph. The optical system of the Soliel saccharimeter is shown in Fig. 139.

N_1 is a polarizing Nicol ; Q a bi-quartz prism ; T the tube containing the solution ; Q_1 is a right-handed quartz plate cut at right angles to the axis ; ABC and DEF are the two wedges of left-handed quartz. N_2 is the " analyser," and L a lower power Galilean telescope which can be focussed on the bi-quartz Q.

When the wedges are set at zero, they with Q_1 produce no effect, and the tint of passage is seen on both sides of the bi-quartz. When an " active " solution is inter-

posed in T, the change in tints is neutralized by bringing the wedges into play. N and Q_2 are a Nicol and quartz plate to be used if the solution in T is coloured. By their means the light emerging from N_1 is made to be complementary in colour to the solution, and then the appearance is as if the solution were colourless.

APPENDIX

(*a*) THE CLEANING OF OPTICAL SURFACES

THE cleaning of surfaces of optical glass is a subject which cannot be too fully emphasized. Not only is it of importance in the laboratory, but still more so in the optician's assembling or testing room.

One of the best methods of "thoroughly cleaning" an optical surface is to wash it *well* with soap and hot water, using a perfectly clean linen cloth, then rub it well with a cloth dipped in alcohol, finally rinsing it in distilled water and drying with a piece of "grease-free" chamois leather. Great care should be taken not to let the hands or finger-tips come into contact with any surface; it will be found advisable to wear a pair of chamois leather gloves when cleaning.

When mounting optical work into instruments it will be found advantageous to immerse the glass in a 20 per cent. solution of nitric acid for about two hours before the cleaning (as mentioned above) is begun, as this prevents to some extent the very objectionable "filming" that occurs on the optical surfaces when optical work remains in an instrument for some considerable period. In instruments that are finally sealed and made air-tight it is advisable to do all mounting in a perfectly dry atmosphere. All particles of dust should be removed with a small camel-hair brush. Such brushes should be continually washed out in distilled water to prevent grease clinging to the small hairs.

For surfaces of ordinary glass (*i.e.* non-optical) a paste, made up of "rouge and ammonia," serves extremely well for cleaning purposes, and should be applied with a piece of chamois leather or a "Selvyt" cloth.

The "pith" in sticks of "elder" are very useful for removing "tarnish" from surfaces of the denser flint glasses.

(*b*) SILVERING OF GLASS

In silvering, cleanliness is again the all-important factor for success.

First of all prepare two solutions :

1. Dissolve silver nitrate in distilled water, and add ammonia till the precipate first thrown down is almost entirely redissolved. Filter the solution, and dilute it so that 100 c.cs. contain 1 gramme of silver nitrate.
2. Dissolve 2 grammes of silver nitrate in a little distilled water and pour it into a litre of boiling distilled water. Add 1·6 grammes of Rochelle salt, and boil the mixture for a short time, till the precipitate contained in it becomes grey ; filter the solution whilst it is still hot.

The glass should then be "*thoroughly*" cleaned, with the same precautions taken as mentioned in the previous section, and whilst still wet from the lastly applied distilled water, should be placed in a *clean* glass vessel (*e.g.* a crystallizing dish), with the surface to be silvered placed uppermost.

Equal quantities of the solutions 1 and 2 should then be mixed together and poured into the vessel so as to cover the glass,—the solutions should be cold. After about an hour the silvering will be completed. The liquid can then be poured off and the glass removed ; any of the silver deposit can be rubbed off where it is not required, and that which is required may be coated with some black varnish for preserving purposes when the silver has dried.

(*c*) GRINDING AND POLISHING A FLAT GLASS SURFACE

The fact of being able to grind and polish a flat surface on a piece of glass is of great importance both for instruc-

tional purposes in the laboratory and for commercial purposes in the workshop.

Such a subject is of too large a scope to deal with very fully in these pages, as practical experience is the chief key to success; but a general outline of the methods employed will no doubt be of use.

It will be presumed that some sort of machine for revolving the tools is available, either the treadle type of "grinder" or the type fitted with a small power unit.

First of all screw the "roughing tool" to the spindle of the machine, and take a little emery (about grade 90 *), mix it with water, and use a little at a time on the tool. Hold the piece of glass in the fingers of both hands firmly, and revolving the "rougher," press the glass down on the tool, giving it a backward and forward motion. In due time all the prominent irregularities of the glass surface will be removed and a smooth ground surface will be left. Another tool for finer grinding is now used. This tool should have already been made a fairly correct flat surface, and therefore may be used for the more exact work. "Fine grinding" can then be done by using 10-minute,* 15-minute, 20-minute, and 60-minute emery in succession in the same way, the grinding being continued with each grade until all bits and scratches left from the coarser grades are removed. The surface must be continually viewed with a fairly high-power eyepiece in order to detect such scratches.

As each grade of emery is used care must be taken to remove any particles of a previous grade; this is best done with a small soft sponge, and by rubbing a rough piece of flat glass known as a "bruiser" on the tool prior to using the actual glass surface on the tool.

When the surface has been successfully brought to the finest condition, the polisher may then be prepared.

* The numbered grades of emery, such as 90, refer to the number of meshes per inch of a sieve through which that particular emery has passed. The 10, 20, 60, etc., minute emery refer to the particles that are left in suspension in water after having been allowed to stand for the respective number of minutes.

Preparing the Polisher.—The polisher is made up of a layer of pitch melted on to the surface of one of the iron tools. The pitch, which may be softened by the addition of tallow or lard, is freed from grit by straining it through a piece of fine muslin on to the tool while the pitch is molten and hot. The tool is heated sufficiently to keep the pitch plastic, and its surface is then flattened by pressing the pitch down on a cold iron plate. Before the pitch is quite hard a number of grooves may be cut in it, in order to give places which will accumulate the polishing medium.

The polishing can then be commenced. The "polisher" should be screwed to the spindle of the machine, warmed slightly, and moistened with a little "rouge and water." The glass surface should be rubbed over the "polisher" as during the grinding process, but in this case the speed of relative movement between glass surface and polisher should be very much slower.

After about twenty minutes' polishing the glass surface will be ready for the "test plate." This is placed on the surface and the interference fringes viewed by reflected light from a Mercury Vapour Lamp; from this is ascertained the relative roundness of the surface and its form, whether convex or concave.

If the surface is convex, the best procedure to try and correct this tendency is to increase the "stroke" and to press harder on the polisher. The result will be to increase the wearing of the surface in the centre and thus give a tendency towards concavity. If the surface is concave, however, the stroke should be shortened and some of the pressure on the tool relaxed. There are various ways of varying the relative amounts of wear in different regions of the surface, such as cutting grooves in certain parts of the polisher to alter the glass surface in the same part; but experience is the only master which can teach all the devices used in practice for the correction of surfaces in such a manner.

The period necessary to complete the polishing will,

of course, depend on the time taken entirely to remove all trace of "grey" from the surface and to produce the best flat possible.

(*d*) **BALSAMING**

When balsaming it is of first importance that the surfaces to be put in contact are absolutely clean and "dust-free." The surfaces should be cleaned as mentioned in section (*a*) and carefully dusted with a soft camel-hair brush. All "balsam" should be carefully filtered before use.

The two optical parts which are to be cemented together should first be slightly warmed in the balsaming oven. A very suitable oven for this purpose is the small (9 in. cube) copper oven supplied by Messrs Baird & Tatlock of Hatton Garden, and is fitted with gas heating. Failing this, an ordinary biscuit tin may be converted into an oven, the heating being provided by a carbon filament electric lamp in the circuit of which is arranged a variable resistance. The lamp should be placed inside the tin, and means for fitting a thermometer and adjustable air regulation provided in the lid. Such a device works extremely well.

A small amount of balsam should then be placed in the centre of one of the surfaces which is to be balsamed, and the other surface pressed carefully but firmly (with a piece of cork) on to the first until the balsam spreads out as a thin film over the entire surface. Any small bubbles should be removed by pressure with the cork. The parts being balsamed should then be placed on a glass plate covered with paper and supports placed at the sides to prevent any sliding movement of one surface relative to the other. The parts are then put into the oven and the temperature slowly raised until it reaches 77° C., where it should be kept for four hours, and then slowly reduced until the temperature of the room is again attained. The parts can then be removed from the oven and all superfluous balsam cleaned off with benzol. The operation is then complete.

There are various grades of Canada balsam, known as "hard" and "soft" balsam, but all except the "very soft" should be taken to 77° C. The "very soft" will be sufficiently mobile to be put on without any heat and

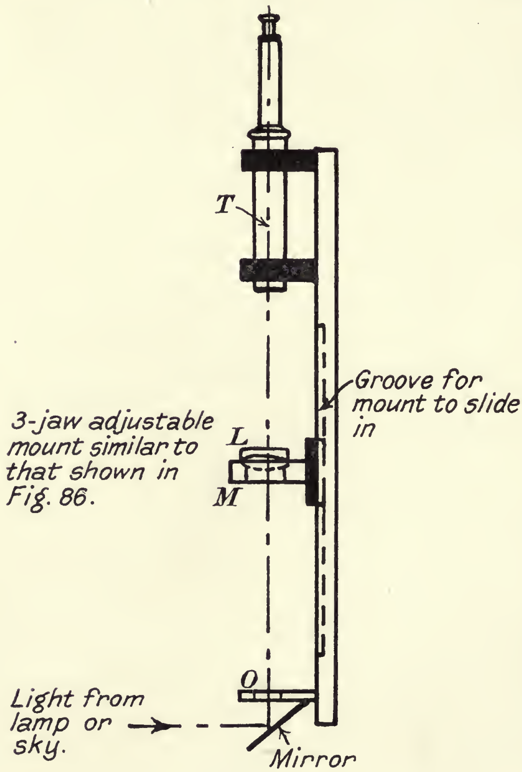


Fig. 140.

will set when left exposed to the atmosphere for an hour or two.

When achromatic lenses are being balsamed it is necessary to "centre" the two lenses while the balsam is still "plastic." For this purpose a piece of apparatus similar to that shown in Fig. 140 will be found of great convenience. It consists of a cross-line object O, an adjustable mount M for the lens L, and a telescope T, all

mounted on the same rigid base and supported in a vertical position.* The lens is rested in the recessed mount M, which is adjusted so that O is in the focal plane of the lens. Observing through the telescope the lens is then rotated, when any centring defect will be shown up by movement of the image. The lens and lens mount can then be heated while in this position until the balsam becomes sufficiently plastic to move one lens relative to the other, when the test can again be repeated until the centring is correct.

(*e*) DEVELOPERS FOR PHOTOGRAPHIC WORK

Hydroquinone Developer

For Plates

Solutions A and B to be mixed in equal quantities when required for use. They should be *kept* in separate bottles.

Solution A.

Hydroquinone	25 gms.
Potassium Metabisulphite . . .	25 gms.
Potassium Bromide	12 gms.
Water	1000 c.cs.

Solution B.

Potassium Hydrate	150 gms.
Water	1000 c.cs.

Fixing Bath

Hypo	150 gms.
Water	1000 c.cs.

* The axis of cross-line and positive component of the achromatic lens is first set co-linear with the axis of the telescope, before the negative or flint component is put on to the first.

Pyro Developer

For Plates

Solutions A and B to be mixed in equal quantities when required for use. They should be *kept* in separate bottles.

Solution A.

Pyrogallic Acid	10 gms.
Potassium Metabisulphite . . .	2·4 gms.
Water	1000 c.cs.

Solution B.

Sodium Carbonate	100 gms.
Sodium Sulphite	100 gms.
Potassium Bromide	12 gms.
Water	1000 c.cs.

Developer

For Gaslight Paper

Sodium Carbonate	170 gms.
Sodium Sulphite	30 gms.
Hydroquinone	8 gms.
Metol	2·5 gms.
Potassium Bromide	1 gm.
Water	1000 c.cs.

(i) A FROSTING SOLUTION FOR GLASS

Such a solution is very convenient for frosting electric lamp bulbs, instead of using tissue paper over a bulb, a much practised method in opticians' workshops.

Dissolve :

25 grammes of (leaf) gelatine and 120 grammes of either calcium carbonate or magnesium oxide in 250 c.cs. of hot distilled water.

Let the solution cool to 34° C. and dip the glass into

it. Allow to dry and then immerse the glass a second time.

Two coats will in general be enough, but more may be given if required.

(ii) **A CEMENT FOR OPTICAL PURPOSES**

For cementing glass cells, glass windows to metal cells, etc., etc., one of the best cements will be found by mixing equal quantities of "*beeswax*" and "*rosin*" (whilst molten), and on cooling make it into thin "sticks." It should be applied with a small heated rod, and then, placing all parts to be cemented into a hot-air oven, should be left until the cement becomes "plastic." At this stage the required surfaces should be put in contact, and then allowed to cool.

This cement will resist the action of aqueous solutions and organic solvents for a very considerable time.

(*f*) TABLE OF USEFUL WAVE-LENGTHS

Substance.	How emitted.	Wave-length in 10^{-8} cms.	Colour.
Sodium	Bunsen Flame	5890·2	Orange
,,	,,	5896·2	,,
Lithium	On pole of "Arc"	6708·2	Red
Rubidium	,,	7947·0	Far red
,,	,,	7806·1	,,
Hydrogen	Vacuum Tube	6563·0	Red
,,	,,	4861·5	Blue-green
,,	,,	4340·7	Violet
Mercury	Mercury Lamp	5790·7	Yellow
,,	,,	5769·6	,,
,,	,,	5460·7	Green
,,	,,	4078·1	Violet
Cadmium	Vacuum Tube	6438·5	Red
,,	,,	5085·8	Green
,,	,,	4799·9	Blue
Strontium	Bunsen Flame	4607·5	Blue

REFRACTIVE INDICES FOR SODIUM LIGHT ($\lambda=589\ \mu\mu$)

Substance.	Refractive Index.
Fluorspar	1·4339
Quartz	1·5442 ordinary
,,	1·5533 extraordinary
Rocksalt	1·5443
Water	1·3329
Carbon Bisulphide	1·6277
Benzene	1·5004
Iceland Spar	1·6584 ordinary
,,	1·4864 extraordinary

TABLES

	0	1	2	3	4	5	6	7	8	9	Mean Differences. 1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	2	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	2	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	Mean Differences. 1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences. 1′	2′	3′	4′	5′
0°	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	14
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10°	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20°	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30°	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40°	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
											1′	2′	3′	4′	5′
45°	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60°	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70°	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80°	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0

N.B.—Subtract Mean Differences.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
											1′	2′	3′	4′	5′
0°	1·000	1·000	1·000	1·000	1·000	1·000	·9999	9999	9999	9999	0	0	0	0	0
1	·9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	·9994	9993	9993	9992	9991	9990	9990	9989	9988	9987	0	0	0	1	1
3	·9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	·9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	·9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	2
6	·9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	·9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	·9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	·9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10°	·9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	·9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	·9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	·9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	·9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	·9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	·9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	·9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	3	4
18	·9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	·9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20°	·9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	·9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	·9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	·9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	·9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	·9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	·8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	·8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	·8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	·8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30°	·8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	·8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	·8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	·8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	·8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	·8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	·8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	·7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	·7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	·7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40°	·7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	·7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	·7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	·7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	·7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

N.B.—Subtract Mean Differences.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences. 1′	2′	3′	4′	5′
45°	·7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	·6947	6934	6921	6099	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	·6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	·6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	·6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50°	·6428	6414	4061	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	·6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	·6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	·6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	·5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	·5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	·5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	·5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60°	·5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	·4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	·4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	·4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	·4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	14
67	·3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	14
68	·3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	·3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70	·3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	·3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	·2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	·2756	2740	2723	2706	2689	2672	2656	2639	2662	2605	3	6	8	11	14
75	·2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	·2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	·2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	·2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	·1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80°	·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	12	14
81	·1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	·1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	·1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	·0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	14
86	·0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	·0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences. 1′	2′	3′	4′	5′
0°	0·000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10°	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20°	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30°	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40°	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences. 1′	2′	3′	4′	5′
45°	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1.0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50°	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60°	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	8375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	0057	$\bar{0}145$	$\bar{0}233$	$\bar{0}323$	$\bar{0}413$	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70°	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2·9042	9208	9375	9544	9714	9887	$\bar{0}061$	$\bar{0}237$	$\bar{0}415$	$\bar{0}595$	29	58	87	116	145
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972					
77	4·3315	3662	4015	4374	4737	5107	5483	5864	6252	6646					
78	4·7046	7453	7867	8288	8716	9152	9594	$\bar{0}045$	$\bar{0}504$	$\bar{0}970$					
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80°	5·6713	7297	7894	8502	9124	9758	$\bar{0}405$	$\bar{1}066$	$\bar{1}742$	$\bar{2}432$					
81	6·3128	3859	4596	5350	6122	6912	7720	8548	9395	$\bar{0}264$	*Mean* differences no longer sufficiently accurate.				
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	$\bar{0}285$					
83	8·1443	2636	3863	5126	6427	7769	9152	$\bar{0}579$	$\bar{2}052$	$\bar{3}572$					
84	9·514	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences. 1′	2′	3′	4′	5′
0°	$-\infty$	7·2419	5429	7190	8439	9408	$\bar{0}$200	$\bar{0}$870	$\bar{1}$450	$\bar{1}$961					
1	8·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	8·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	8·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326					
4	8·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	8·9403	9489	9573	9655	9736	9816	9894	9970	$\bar{0}$046	$\bar{0}$120	13	26	39	52	65
6	9·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	9·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	9·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	9·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10°	9·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	9·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	9·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	9·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	9·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	9·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	9·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	9·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	9·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	9·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20°	9·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	9·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	9·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	9·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	9·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	9·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	9·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	9·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	9·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	9·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30°	9·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	9·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	9·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	9·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	9·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	9·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	9·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	9·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	9·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	9·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40°	9·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	9·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	9·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	9·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	9·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences. 1′	2′	3′	4′	5′
45°	9·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	9·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	9·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	9·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	9·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50°	9·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	9·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	9·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	9·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	9·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	9·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	9·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	9·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	9·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	9·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60°	9·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	9·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	9·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	9·9499	9503	9507	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	9·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	9·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	9·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	9·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	9·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	9·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70°	9·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	9·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	9·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	9·9806	9808	9811	9813	9815	9817	9820	8922	9824	9826	0	1	1	2	2
74	9·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	9·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	9·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	9·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	9·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	9·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80°	9·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	9·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	9·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83	9·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	0	1	1
84	9·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85	9·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86	9·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87	9·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88	9·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89	9·9999	9999	$\bar{0}000$	$\bar{0}000$	$\bar{0}000$	$\bar{0}000$	$\bar{0}000$	$\bar{0}000$	$\bar{0}000$	$\bar{0}000$	0	0	0	0	0

Subtract Mean Differences.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences. 1′	2′	3′	4′	5′
0°	10·0000	0000	0000	0000	0000	0000	0000	0000	0000	9.9999	0	0	0	0	0
1	9·9999	9999	9999	9999	9999	9999	9998	9998	9998	9998	0	0	0	0	0
2	9·9997	9997	9997	9996	9996	9996	9996	9995	9995	9994	0	0	0	0	0
3	9·9994	9994	9993	9993	9992	9992	9991	9991	9990	9990	0	0	0	0	0
4	9·9989	9989	9988	9988	9987	9987	9986	9985	9985	9984	0	0	0	0	0
5	9·9983	9983	9982	9981	9981	9980	9979	9978	9978	9977	0	0	0	0	1
6	9·9976	9975	9975	9974	9973	9972	9971	9970	9969	9968	0	0	0	1	1
7	9·9968	9967	9966	9965	9964	9963	9962	9961	9960	9959	0	0	1	1	1
8	9·9958	9956	9955	9954	9953	9952	9951	9950	9949	9947	0	0	1	1	1
9	9·9946	9945	9944	9943	9941	9940	9939	9937	9936	9935	0	0	1	1	1
10°	9·9934	9932	9931	9929	9928	9927	9925	9924	9922	9921	0	0	1	1	1
11	9·9919	9918	9916	9915	9913	9912	9910	9909	9907	9906	0	1	1	1	1
12	9·9904	9902	9901	9899	9897	9896	9894	9892	9891	9889	0	1	1	1	1
13	9·9887	9885	9884	9882	9880	9878	9876	9875	9873	9871	0	1	1	1	2
14	9·9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15	9·9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16	9·9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	2	2
17	9·9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18	9·9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19	9·9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20°	9·9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21	9·9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22	9·9672	9669	9666	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23	9·9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24	9·9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25	9·9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26	9·9537	9533	9529	9525	9522	9518	9514	9510	9507	9503	1	1	2	3	3
27	9·9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28	9·9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29	9·9418	9414	9410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30°	9·9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31	9·9331	9326	9322	9317	9312	9308	9303	9298	9294	9289	1	2	2	3	4
32	9·9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33	9·9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34	9·9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35	9·9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36	9·9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37	9·9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38	9·8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39	9·8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40°	9·8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41	9·8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	5	6
42	9·8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43	9·8641	8634	8627	8620	8613	8606	8598	8591	8584	8577	1	2	4	5	6
44	9·8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6

Subtract Mean Differences.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences. 1′	2′	3′	4′	5′
45°	9·8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5	6
46	9·8418	8410	8402	8394	8386	8378	8370	8362	8354	8346	1	3	4	5	7
47	9·8338	8330	8322	8313	8305	8297	8289	8280	8272	8264	1	3	4	6	7
48	9·8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6	7
49	9·8169	8161	8152	8143	8134	8125	8117	8108	8099	8090	1	3	4	6	7
50°	9·8081	8072	8063	8053	8044	8035	8026	8017	8007	7998	2	3	5	6	8
51	9·7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6	8
52	9·7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7	8
53	9·7795	7785	7774	7764	7754	7744	7734	7723	7713	7703	2	3	5	7	9
54	9·7692	7682	7671	7661	7650	7640	7629	7618	7607	7597	2	4	5	7	9
55	9·7586	7575	7564	7553	7542	7531	7520	7509	7498	7487	2	4	6	7	9
56	9·7476	7464	7453	7442	7430	7419	7407	7396	7384	7373	2	4	6	8	10
57	9·7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8	10
58	9·7242	7230	7218	7205	7193	7181	7168	7156	7144	7131	2	4	6	8	10
59	9·7118	7106	7093	7080	7068	7055	7042	7029	7016	7003	2	4	6	9	11
60°	9·6990	6977	6963	6950	6937	6923	6910	6896	6883	6869	2	4	7	9	11
61	9·6856	6842	6828	6814	6801	6787	6773	6759	6744	6730	2	5	7	9	12
62	9·6716	6702	6687	6673	6659	6644	6629	6615	6600	6585	2	5	7	10	12
63	9·6570	6556	6541	6526	6510	6495	6480	6465	6449	6434	3	5	8	10	13
64	9·6418	6403	6387	6371	6356	6340	6324	6308	6292	6276	3	5	8	11	13
65	9·6259	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11	14
66	9·6093	6076	6059	6042	6024	6007	5990	5972	5954	5937	3	6	9	12	15
67	9·5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12	15
68	9·5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13	16
69	9·5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14	17
70°	9·5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14	18
71	9·5126	5104	5082	5060	5037	5015	4992	4969	4946	4923	4	8	11	15	19
72	9·4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16	20
73	9·4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	4	9	13	17	21
74	9·4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	5	9	14	18	23
75	9·4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	5	10	15	20	24
76	9·3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	5	11	16	21	26
77	9·3521	3488	3455	3421	3387	3353	3319	3284	3250	3214	6	11	17	23	28
78	9·3179	3143	3107	3070	3034	2997	2959	2921	2883	2845	6	12	19	25	31
79	9·2806	2767	2727	2687	2647	2606	2565	2524	2482	2439	7	14	20	27	34
80°	9·2397	2353	2310	2266	2221	2176	2131	2085	2038	1991	8	15	23	30	38
81	9·1943	1895	1847	1797	1747	1697	1646	1594	1542	1489	8	17	25	34	42
82	9·1436	1381	1326	1271	1214	1157	1099	1040	0981	0920	10	19	29	38	48
83	9·0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	11	22	33	44	55
84	9·0192	0120	0046	$\overline{9}970$	$\overline{9}894$	$\overline{9}816$	$\overline{9}736$	$\overline{9}655$	$\overline{9}573$	$\overline{9}489$	13	26	39	52	65
85	9·9403	9315	9226	9135	9042	8946	8849	8749	8647	8543	16	32	48	64	80
86	8·8436	8326	8213	8098	7979	7857	7731	7602	7468	7330					
87	8·7188	7041	6889	6731	6567	6397	6220	6035	5842	5640					
88	8·5428	5206	4971	4723	4459	4179	3880	3558	3210	2832					
89	8·2419	1961	1450	0870	0200	$\overline{9}408$	$\overline{8}439$	$\overline{7}190$	$\overline{5}429$	$\overline{2}419$					

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences. 1′	2′	3′	4′	5′
0°	−∞	7.2419	5429	7190	8439	9409	$\bar{0}200$	$\bar{0}370$	$\bar{1}450$	$\bar{1}962$					
1	8·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	8·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3	8·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336					
4	8·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5	8·9420	9506	9591	9674	9756	9836	9915	9992	$\bar{0}068$	$\bar{0}143$	13	26	40	53	66
6	9·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	9·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	9·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	9.1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10°	9·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	9·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	9·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	9·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	9·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	9·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	9·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	9·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	9·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	9·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20°	9·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	9·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	9·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	9·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	9·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	9·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	9·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	9·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	9·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	9·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30°	9·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	9·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	9·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	9·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	9·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	9·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	9·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	9·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	9·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	9·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40°	9·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	9·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	9·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	9·9697	9712	9727	9742	9757	9773	9788	9803	9818	9833	3	5	8	10	13
44	9·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences. 1′	2′	3′	4′	5′
45°	10·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46	10·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47	10·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48	10·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49	10·0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50°	10·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51	10·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52	10·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53	10·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54	10·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	3	5	8	11	13
55	10·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56	10·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57	10·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58	10·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59	10·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60°	10·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61	10·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62	10·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63	10·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64	10·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	6	10	13	16
65	10·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66	10·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67	10·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68	10·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	19
69	10·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70°	10·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71	10·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72	10·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73	10·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74	10·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75	10·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76	10·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77	10·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78	10·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79	10·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80°	10·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8	16	23	31	39
81	10·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82	10·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
83	10·9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	11	12	34	45	56
84	10·9784	9857	9932	$\bar{0}$008	$\bar{0}$085	$\bar{0}$164	$\bar{0}$244	$\bar{0}$326	$\bar{0}$409	$\bar{0}$494	13	26	40	53	66
85	11·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	16	32	48	64	81
86	11·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663					
87	11·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357					
88	11·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89	11·7581	8038	8550	9130	9800	$\bar{0}$591	$\bar{1}$561	$\bar{2}$810	$\bar{4}$571	$\bar{7}$581					

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	10000	9901	9804	9709	9615	9524	9434	9346	9259	9174	9	18	27	36	45	55	64	73	82
11	9091	9009	8929	8850	8772	8696	8621	8547	8475	8403	8	15	23	30	38	45	53	61	68
12	8333	8264	8197	8130	8065	8000	7937	7874	7813	7752	6	13	19	26	32	38	45	51	58
13	7692	7634	7576	7519	7463	7407	7353	7299	7246	7194	5	11	16	22	27	33	38	44	49
14	7143	7092	7042	6993	6944	6897	6849	6803	6757	6711	5	10	14	19	24	29	33	38	43
15	6667	6623	6579	6536	6494	6452	6410	6369	6329	6289	4	8	13	17	21	25	29	33	38
16	6250	6211	6173	6135	6098	6061	6024	5988	5952	5917	4	7	11	15	18	22	26	29	33
17	5882	5848	5814	5780	5747	5714	5682	5650	5618	5587	3	6	10	13	16	20	23	26	29
18	5556	5525	5495	5464	5435	5405	5376	5348	5319	5291	3	6	9	12	15	17	20	23	26
19	5263	5236	5208	5181	5155	5128	5102	5076	5051	5025	3	5	8	11	13	16	18	21	24
20	5000	4975	4950	4926	4902	4878	4854	4831	4808	4785	2	5	7	10	12	14	17	19	21
21	4762	4739	4717	4695	4673	4651	4630	4608	4587	4566	2	4	7	9	11	13	15	17	20
22	4545	4525	4505	4484	4464	4444	4425	4405	4386	4367	2	4	6	8	10	12	14	16	18
23	4348	4329	4310	4292	4274	4255	4237	4219	4202	4184	2	4	5	7	9	11	13	14	16
24	4167	4149	4132	4115	4098	4082	4065	4049	4032	4016	2	3	5	7	8	10	12	13	15
25	4000	3984	3968	3953	3937	3922	3906	3891	3876	3861	2	3	5	6	8	9	11	12	14
26	3846	3831	3817	3802	3788	3774	3759	3745	3731	3717	1	3	4	6	7	8	10	11	13
27	3704	3690	3676	3663	3650	3636	3623	3610	3597	3584	1	3	4	5	7	8	9	11	12
28	3571	3559	3546	3534	3521	3509	3497	3484	3472	3460	1	2	4	5	6	7	9	10	11
29	3448	3436	3425	3413	3401	3390	3378	3367	3356	3344	1	2	3	5	6	7	8	9	10
30	3333	3322	3311	3300	3289	3279	3268	3257	3247	3236	1	2	3	4	5	6	7	9	10
31	3226	3215	3205	3195	3185	3175	3165	3155	3145	3135	1	2	3	4	5	6	7	8	9
32	3125	3115	3106	3096	3086	3077	3067	3058	3049	3040	1	2	3	4	5	6	7	8	9
33	3030	3021	3012	3003	2994	2985	2976	2967	2959	2950	1	2	3	4	4	5	6	7	8
34	2941	2933	2924	2915	2907	2899	2890	2882	2874	2865	1	2	3	3	4	5	6	7	8
35	2857	2849	2841	2833	2825	2817	2809	2801	2793	2786	1	2	3	3	4	5	6	6	7
36	2778	2770	2762	2755	2747	2740	2732	2725	2717	2710	1	2	2	3	4	5	5	6	7
37	2703	2695	2688	2681	2674	2667	2660	2653	2646	2639	1	1	2	3	4	4	5	6	6
38	2632	2625	2618	2611	2604	2597	2591	2584	2577	2571	1	1	2	3	3	4	5	5	6
39	2564	2558	2551	2545	2538	2532	2525	2519	2513	2506	1	1	2	3	3	4	4	5	6
40	2500	2494	2488	2481	2475	2469	2463	2457	2451	2445	1	1	2	2	3	4	4	5	5
41	2439	2433	2427	2421	2415	2410	2404	2398	2392	2387	1	1	2	2	3	3	4	5	5
42	2381	2375	2370	2364	2358	2353	2347	2342	2336	2331	1	1	2	2	3	3	4	4	5
43	2326	2320	2315	2309	2304	2299	2294	2288	2283	2278	1	1	2	2	3	3	4	4	5
44	2273	2268	2262	2257	2252	2247	2242	2237	2232	2227	1	1	2	2	3	3	4	4	5
45	2222	2217	2212	2208	2203	2198	2193	2188	2183	2179	0	1	1	2	2	3	3	4	4
46	2174	2169	2165	2160	2155	2151	2146	2141	2137	2132	0	1	1	2	2	3	3	4	4
47	2128	2123	2119	2114	2110	2105	2101	2096	2092	2088	0	1	1	2	2	3	3	4	4
48	2083	2079	2075	2070	2066	2062	2058	2053	2049	2045	0	1	1	2	2	3	3	3	4
49	2041	2037	2033	2028	2024	2020	2016	2012	2008	2004	0	1	1	2	2	2	3	3	4
50	2000	1996	1992	1988	1984	1980	1976	1972	1969	1965	0	1	1	2	2	2	3	3	4
51	1961	1957	1953	1949	1946	1942	1938	1934	1931	1927	0	1	1	2	2	2	3	3	3
52	1923	1919	1916	1912	1908	1905	1901	1898	1894	1890	0	1	1	1	2	2	3	3	3
53	1887	1883	1880	1876	1873	1869	1866	1862	1859	1855	0	1	1	1	2	2	2	3	3
54	1852	1848	1845	1842	1838	1835	1832	1828	1825	1821	0	1	1	1	2	2	2	3	3

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	1818	1815	1812	1808	1805	1802	1799	1795	1792	1789	0	1	1	1	2	2	2	3	3
56	1786	1783	1779	1776	1773	1770	1767	1764	1761	1757	0	1	1	1	2	2	2	3	3
57	1754	1751	1748	1745	1742	1739	1736	1733	1730	1727	0	1	1	1	2	2	2	2	3
58	1724	1721	1718	1715	1712	1709	1706	1704	1701	1698	0	1	1	1	1	2	2	2	3
59	1695	1692	1689	1686	1684	1681	1678	1675	1672	1669	0	1	1	1	1	2	2	2	3
60	1667	1664	1661	1658	1656	1653	1650	1647	1645	1642	0	1	1	1	1	2	2	2	3
61	1639	1637	1634	1631	1629	1626	1623	1621	1618	1616	0	1	1	1	1	2	2	2	2
62	1613	1610	1608	1605	1603	1600	1597	1595	1592	1590	0	1	1	1	1	2	2	2	2
63	1587	1585	1582	1580	1577	1575	1572	1570	1567	1565	0	0	1	1	1	1	2	2	2
64	1563	1560	1558	1555	1553	1550	1548	1546	1543	1541	0	0	1	1	1	1	2	2	2
65	1538	1536	1534	1531	1529	1527	1524	1522	1520	1517	0	0	1	1	1	1	2	2	2
66	1515	1513	1511	1508	1506	1504	1502	1499	1497	1495	0	0	1	1	1	1	2	2	2
67	1493	1490	1488	1486	1484	1481	1479	1477	1475	1473	0	0	1	1	1	1	2	2	2
68	1471	1468	1466	1464	1462	1460	1458	1456	1453	1451	0	0	1	1	1	1	2	2	2
69	1449	1447	1445	1443	1441	1439	1437	1435	1433	1431	0	0	1	1	1	1	2	2	2
70	1429	1427	1425	1422	1420	1418	1416	1414	1412	1410	0	0	1	1	1	1	1	2	2
71	1408	1406	1404	1403	1401	1399	1397	1395	1393	1391	0	0	1	1	1	1	1	2	2
72	1389	1387	1385	1383	1381	1379	1377	1376	1374	1372	0	0	1	1	1	1	1	2	2
73	1370	1368	1366	1364	1362	1361	1359	1357	1355	1353	0	0	1	1	1	1	1	2	2
74	1351	1350	1348	1346	1344	1342	1340	1339	1337	1335	0	0	1	1	1	1	1	1	2
75	1333	1332	1330	1328	1326	1325	1323	1321	1319	1318	0	0	1	1	1	1	1	1	2
76	1316	1314	1312	1311	1309	1307	1305	1304	1302	1300	0	0	1	1	1	1	1	1	2
77	1299	1297	1295	1294	1292	1290	1289	1287	1285	1284	0	0	0	1	1	1	1	1	1
78	1282	1280	1279	1277	1276	1274	1272	1271	1269	1267	0	0	0	1	1	1	1	1	1
79	1266	1264	1263	1261	1259	1258	1256	1255	1253	1252	0	0	0	1	1	1	1	1	1
80	1250	1248	1247	1245	1244	1242	1241	1239	1238	1236	0	0	0	1	1	1	1	1	1
81	1235	1233	1232	1230	1229	1227	1225	1224	1222	1221	0	0	0	1	1	1	1	1	1
82	1220	1218	1217	1215	1214	1212	1211	1209	1208	1206	0	0	0	1	1	1	1	1	1
83	1205	1203	1202	1200	1199	1198	1196	1195	1193	1192	0	0	0	1	1	1	1	1	1
84	1190	1189	1188	1186	1185	1183	1182	1181	1179	1178	0	0	0	1	1	1	1	1	1
85	1176	1175	1174	1172	1171	1170	1168	1167	1166	1164	0	0	0	1	1	1	1	1	1
86	1163	1161	1160	1159	1157	1156	1155	1153	1152	1151	0	0	0	1	1	1	1	1	1
87	1149	1148	1147	1145	1144	1143	1142	1140	1139	1138	0	0	0	1	1	1	1	1	1
88	1136	1135	1134	1133	1131	1130	1129	1127	1126	1125	0	0	0	1	1	1	1	1	1
89	1124	1122	1121	1120	1119	1117	1116	1115	1114	1112	0	0	0	1	1	1	1	1	1
90	1111	1110	1109	1107	1106	1105	1104	1103	1101	1100	0	0	0	1	1	1	1	1	1
91	1099	1098	1096	1095	1094	1093	1092	1091	1089	1088	0	0	0	0	1	1	1	1	1
92	1087	1086	1085	1083	1082	1081	1080	1079	1078	1076	0	0	0	0	1	1	1	1	1
93	1075	1074	1073	1072	1071	1070	1068	1067	1066	1065	0	0	0	0	1	1	1	1	1
94	1064	1063	1062	1060	1059	1058	1057	1056	1055	1054	0	0	0	0	1	1	1	1	1
95	1053	1052	1050	1049	1048	1047	1046	1045	1044	1043	0	0	0	0	1	1	1	1	1
96	1042	1041	1040	1038	1037	1036	1035	1034	1033	1032	0	0	0	0	1	1	1	1	1
97	1031	1030	1029	1028	1027	1026	1025	1024	1022	1021	0	0	0	0	1	1	1	1	1
98	1020	1019	1018	1017	1016	1015	1014	1013	1012	1011	0	0	0	0	1	1	1	1	1
99	1010	1009	1008	1007	1006	1005	1004	1003	1002	1001	0	0	0	0	0	1	1	1	1